TITLE:

Application of topology to bandwidth reduction method of structural stiffness matrix(Dissertation_全文)

AUTHOR(S):

Taniguchi, Takeo

CITATION:

Taniguchi, Takeo. Application of topology to bandwidth reduction method of structural stiffness matrix. 京都大学, 1975, 工学博士

ISSUE DATE:

1975-03-24

URL:

https://doi.org/10.14989/doctor.k1600

RIGHT:

京都大学学術情報リポジトリ
KURENAI 紅
Kyoto University Research Information Repository

APPLICATION OF TOPOLOGY TO BANDWIDTH REDUCTION METHOD OF STRUCTURAL STIFFNESS MATRIX

By

Takeo TANIGUCHI

Department of Civil Engineering
Kyoto University
Kyoto, Japan

December, 1974

APPLICATION OF TOPOLOGY TO BANDWIDTH REDUCTION METHOD OF STRUCTURAL STIFFNESS MATRIX

By

Takeo TANIGUCHI

Department of Civil Engineering
Kyoto University
Kyoto, Japan

December, 1974

SUMMARY

This thesis presents some bandwidth reduction methods of stiffness matrix of civil engineering structure. The difference of the structural topology induces the different approaches for obtaining the reduction methods. That is, for a kind of statically determinate system (i.e. tree structure), framed structures or truss-type structures and also for continuous media, the author shows different approaches for the bandwidth reduction methods. One of the largest characteristics of the proposed methods is the introduction of the graph theory. By the aid of the theory the author investigates some factors which are influent on the decision of the minimum value of the bandwidth of any structural system. But they are not fitted as the informations for programming of digital computers. Thus, the proposed methods are the graphical ones and for them the author defines a kind of coordinate systems whose one axis corresponds to the largeness of the bandwidth of any structural system. Any graph corresponding to the model of a structural system is drawn in the coordinate system as to hold the original topology of the system. Thus, the bandwidth reduction method is replaced to how to decrease the length of the graph along the axis in the coordinate system. Furthermore, the nodal-labeling procedure is removed and it is automatically satisfied in the space with the proposed coordinate system. The optimum nodal labeling which expresses the minimum bandwidth of the system is appreciated by the checking procedure whether the length of the drawn graph along the axis can be reduced or not. The optimum state gives the minimum length along the axis and the configuration of the graph in the space is stable. Moreover, the author trys to express the profile of stiffness matrix in the same coordinate system which is proposed for the bandwidth reduction methods. And he indicates the similarity between the bandwidth reduction and profile reduction methods for any structural system.

CONTENTS

PREFACE

In accordance with the development of digital computers man can obtain reasonable solution for problems which were impossible to be solved. But it can be said that the accurate solution can be expected only when the solution can be strictly formulated. Digital computer can lead to the solution which depends on the informations that are formulated in the programming. Thus, lack of important informations for the solution never succeed in using the powerful machine.

In the last decade a number of algorithms for the bandwidth reduction method of stiffness matrix of civil engineering structure were proposed, but the accuracy of the results wholly depends on the given structural configurations. The inspection for those methods finds the lack of the understanding of the characteristic of the machine, that is, in the problem the computer is expected impossible ability for it. In the sense, the bandwidth problem of stiffness matrix should be, once again, examined from the first step. Even if we can find out important informations for the solution, they may be, at this stage, impossible to be formulated. Thus, they cann't be introduced in the form of programming for the machine. They are left as they are, and they should be usefully treated before the stage of programming. According to this thinking the author attacked to the bandwidth problem, and he summarized his results in this thesis. Thus, the results don't come to conflict with the past studies in this field but should be located at the position of the previous stage of them. His results should be introduced before the programmings which were already proposed.

The author would like to notice that his strategy to the problem was much influenced by the theory of catastrophe which was introduced by R. Thom in 1972. He never intend to approach the problem from the algebraic view point but from the graphical one.

It is a great pleasure to make grateful acknowledgement of many helpful suggestions which have been contributed by Professor Ichiro KONISHI and Associate Professor Naruhito SHIRAISHI. To Lecturer Masaru MATSUMOTO the author should express his special thanks for his encouragement, and to Mr. Shinroku YOKOTA he would like to thank for his criticism about the bandwidth reduction method in this thesis. And the author must note that his interest in the field of topology was induced by Mr. Saburo TAMAMURA.

December, 1974

TAKEO TANIGUCHI

CHAPTER 1
INTRODUCTION

1 1. General Remarks

"Given a connected graph with a number of nodes, G(n), which are to be labeled from 1 to n. Minimize the difference of labeled numbers of two nodes which are connected by a line."

The problem seems to be very simple but general solution is not found. The minimum value is decided only by the structure of the graph whose properties are already studied considerably.

Giving physical properties to a graph yields to a network. The application of network analysis to the fields of engineerings has been done and we can find many investigations in the field of structural analysis, too.

The earlier ages of the application of network theory to the strucral analysis were supported by an electric engineer, G. Kron. He analogized and analized any structure by an equivalent electric circuit without using any topological property. The aim of his investigation was to treat huge structures and he proposed "Kron's Piecewise Analysis."[1 5]

The certification of the rightness of Kron's method was done by J. P. Roth who clarified the method by use of topological concepts.[6] And the Householder's formula which is a strictly mathematical method of inverting a matrix shows a general mathematical form of the piecewise method.[7]

In accordance with the development of digital computers a number of structural analyses found the efficiency of the method and the treatment of any structural system as a network was applied not only to the elastic analysis[8 20] but also to the plastic analysis,[20, 21,22] vibrational problems,[1] structural designs,[21,23] and et al.[24,25] That is, the concept of the piecewise method can, to some extent, reduce difficulties which appear by use of the displacement method that fits to the characteristic of computers. The piecewise method can save necessary memories and the running-time of computer for the inversion procedure of any structural stiffness matrix, because certain parts of zero elements in the matrix which are useless and occupy a large part of memory core are not taken into consideration and are ignored by use of the method. But a stiffiness matrix of each sub-system does necessarily contain a lot of zeros.

In general a stiffness matrix has a distinguished property that non-zero elements can be gathered within some lines along the principal diagonal line of the matrix. Thus, the piecewise method does not use the usefull property for every stiffness matrix of subdivided system.

The non-zero elements can be gathered near the principal diagonal by proper reordering of nodes in the system. "Band Matrix Method" introduces this property in its algolithm and saves the necessary memory and also the computation time.

The number of lines including all of the non-zero elements is called "Bandwidth" of the matrix and it can be reduced by appropriate reordering of joints. The minimum value of it is one of the characteristics of the system, itself, and it is not influenced by physical properties in the system but it changes sensively by some altering of connections of members. Thus, we can conclude that it is decided only by the topology of the system.

In the study of G. G. Alway and D. W. Martin in 1965 we find following sentence.[31]

> "In the absense of appropriate theorems for the theory of graphs, one approach to the problem is to construct a computer program to survey the n! possible permutations,... "

That is, although they sufficiently appreciated that the topology of structural system must be taken into consideration for the proposal of new bandwidth reduction method, they could not. After the study we can find many methods for bandwidth reduction but most of them didn't introduce any concept of topology or could use only a few of the characteristics even if they could, and depended on some other strategies, for example, the successive permutations of row and column matrices,[31, 32, 33, 41] and the dynamic programming.[34, 35, 36]

The newest algorithm proposed by R. J. Collins seems to be proper from the computational view point, but the running-time varies in accordance with the initial ordering of joints and, at the same time the method has the demerit that it cann't always give the minimum value of bandwidth and it depends on the configuration of the system. It occures from the lack of the introduction of topological property of systems in the proposed method.[37]

In some studies we can find out some topological properties being applied to the bandwidth reduction methods. The method proposed by Kawamo and et al. introduced the concept of "distance" in graph theory, even though they didn't use the word.[38] W. R. Spillers and N. Hickerson used "degree" of joints which is also defined in graph theory.[35] E. Cuthill and J. McKee proposed an algorithm in which "degree" and "distance" were introduced, and their method can give good result for a kind of structure which has a convex boundary configuration.[39, 40]

These methods are effective for the application to some kinds of structural configurations and they can give good results which are near the minimum bandwidth. But, if configurations of systems are not preferable for the methods, they can give insufficient results. That is, the result depends wholly on the configuration of the system being treated.

The matrix analysis of any structure means that the system is transformed into an equivalent network. But the bandwidth problem does not concern with the physical properties in the network and the removal of the properties yields to a graph which consists of only nodes and lines.

Therefore, the bandwidth reduction problem is rewritten as how to minimize the difference of nodal numbers of two nodes which are connected each other by a line. That is, the proposed and unresolved mathematical problem at the beginning of this chapter corresponds to the bandwidth reduction problem. Thus, if the bandwidth reduction is our concern, a new strategy based on the graph theory must be established. Even if new programming for bandwidth reduction is wanted to be produced, new informations from the graph theory must be found and be taken into consideration in the programming.

As mentioned before, the difficulty of the nodal labeling occures from the complex configuration of the structural system. Especially, following two items are the principal reasons for the difficulty.

1. Irregularity of the surrounding configuration.
2. Non-uniformity of nodal distribution in the system.

If the direct labeling of nodes is difficult, the system should be transformed into a new configuration whose nodal labeling can be easily done.

As far as we treat a graph instead of original structure, it can be arbitrarily deformed as we want, though the connectivity relation of lines in original system has to be held.

Thus, it is concluded that the usage of a connected graph is efficient and necessary for the proposal of a new bandwidth reduction method and also that the investigation of topological property in graphs should be done in order to find out some effective factors which give influences on the bandwidth of original stiffness matrix.

The bandwidth reduction theme is how to minimize the maximum difference between two nodal numbers, and how to minimize the total differences between every neighbouring nodal numbers is evidently different from the above theme. This is called "Profile Minimization". The value of "Bandwidth" of "Band Matrix Method" is fixed for any row matrix, and a lot of zeros within the bandwidth may be removed when the bandwidth for every row matrix is changed as to include only these elements which locate between the first nonzero in every row and the diagonal. It indicates the efficiency of the new theme.

For this new theme we have no informations which lead us to its minimum profile, but it is obvious that the value of the profile is decided by the topology of the given system. In the sense, the above considerations for bandwidth reduction are available for this new theme.

1-2. Object and Scope

Main object of this thesis is to propose a new method of bandwidth reduction, but in order to establish the purpose the topological property of a graph has to be sufficiently investigated and what influences on the value of bandwidth must be found.

Even if the mimimum value of bandwidth is one of the characteristics of a graph, it is actually decided by the successive labeling of nodes. If it is proper, the value may show the minimum one. That is, in order to decrease the value to its minimum one all of the nodes must be ordered properly.

As far as an original graph which expresses a structural system is treated, it is often impossible to guess the minimum value of bandwidth, but for some graphs we can easily obtain the minimum ones by mere inspection. Thus, a complex graph which includes two items in previous section should be redrawn into a new configuration from which we can easily guess the bandwidth. In the newly drawing procedure original topology has to be held in the new configuration.

The author proposes a coordinate system where any graph can be newly drawn. From the new configuration in the coordinate system we can easily know the bandwidth and the node-labeling procedure is unnececessary there, for it is automatically labeled as soon as the configuration appears in the coordinate system. That is, in the definition of the coordinate system one axis corresponds to the value of bandwidth and the labeling procedure for nodes is already defined in the coordinate system. This coordinate system is called "A Filing Field". By the introduction of the system the bandwidth reduction method is replaced to how to decrease the value along one axis which corresponds to the value of bandwidth. That is, our concern is how to draw a configuration in the coordinate system as to have the minimum value along the axis. This is the newly proposed strategy for bandwidth reduction method.

Summarizing the new reduction method, it consists of following two procedures;

1. Introduction of a new coordinate system whose one axis presents the bandwidth.
2. Drawing procedure of original graph in the new coodinate system as to have the minimum value along the axis.

The details of the new method are summarized in nine chapters of this thesis. Chapter 2 contributes to clarify the role of the network-topology in stiffness matrix of a structural system and to propose a modification of Kron's piecewise method which can save necessary memory and running-time of computer for inverting procedure of stiffness matrix and fits to analyze a large-scale structure. The proposed method has no necessity to reorder nodes in a system, though the direct application of Kron's method needs the procedure. And in the last the author gives considerations for labeling procedure for piecewise analysis and makes notice that it cann't be done even if the topological property is taken into considerations in the formulation. In Chapter 3 the relations between the topology of structure and its bandwidth are investigated by using a number of actural systems and some important definitions and concepts in the graph theory are introduced. Chapter 4 is used for the definition of the filing field which is a newly proposed coordinate system where original graph is drawn as to express the bandwidth obviously. In Chapter 5, the author proposes an efficient bandwidth reduction method, so called Sequential File Method, which is applied to a kind of statically determinate structures, i.e. tree structures. The possibility of the application of the sequential file method to mesh structures is discussed in Chapter 6 by using a number of examples. Moreover, a bandwidth reduction method for rather simple mesh structures is also proposed. Chapter 7 treates some complicated mesh structures, and effective method for the bandwidth reduction is proposed and is applied to some complicated mesh structural systems as a plate structure with finite delements. In Chapter 8 the author shows that the profile, i.e. the variable bandwidth, for any tree can be decided when the graph is given and he trys to investigate how to decrease the profile of mesh system by the aid of the filing field which is proposed for bandwidth reduction. And concluding remarks are given in the last chapter. There, the author discusses the relations between three methods for the bandwidth reductions which are proposed in this thesis and he examines the merits and the demerits of the methods. And he gives some considerations for the bandwidth reduction method and the profile reduction method from the graph-theoretical view point.

CHAPTER 2
TOPOLOGICAL PROPERTY IN STRUCTURAL STIFFNESS MATRIX

2－1. Introduction

In the field of electric circuit the network-topological formulation was already done and an electric engineer, G. Kron, analogized the structural analysis to the ones of electric circuits.[1-4,12,13]

According to his studies, the displacement method and the force method are equivalent to the node method and the mesh method, respectively. And the third method in the circuit problem i.e. the tree method, is considered as a modification of the displacement method in the structural analysis.

Comparing both methods in structural analysis, the former is generally used in actual computation by use of the digital computers but the method implies the difficulty which occures in treating huge structure. That is, a structure cann't be analyzed at a time if it is large.

One of the efficient methods to overcome the difficulty was proposed by G. Kron and it is called "Pieceweise Method".[1] That is, a huge structure is divided into a number of substructures which can be treated at a time, and combining the results of them yields to the accurate result of the original system.

The network-topological property plays the most important role in the method, because it expresses the connectivity relation of structural elements and the method intends to treat the original system, itself.

The purposes of Kron's method are not only to make the analysis of a huge structure possible but also to save the necessary memory and the running-time of digital computer. His method requires a number of submatrices for the substructures and also a number of stiffness elements corresponding to the connecting members between substructures among all elements of original large stiffness matrix. By use of his method we can remove a lot of zero elements which locate outside of the individual submatrix from the imput data and it induces the saving of the computaion-time.

If the joint-labeling of original system is to be done, the joints in the same subsystem must, at least, be rearranged to be successive each other before the procedure of subdivision.

The rearrangement of joints is easily done by the aid of the figure of original system but if it is proceeded without the original configuration, the nodal rearrangement becomes very difficult. For the purpose of reordering of joints, we need not actual structure but only the graph which shows the connectivity relationship of structural elements.

In this chapter, the investigation of the network-topological property in Kron's method is, at first, done. Followingly, piecewise methods which are induced by the informations of tearing a system obtained from the topological properties are given. And they are compared with Kron's method and the author investigates the characteristics of the proposed methods. In conclusions, the author gives further considerations for more efficient usage of hidden topological property in structural stiffness matrix for the inversion of the matrix.

2-2. Kron's Piecewise Analysis and Householder's Formula

When the physical system cann't be solved as a whole because of the insufficient storage of the computer being used and it takes too much running-time for computation even if possible to calculate, "Kron's Piecewise Analysis" becomes very effective.

His method consists of three main steps;[1-4, 12,13]

1. Tear apart the given physical system into a convienient number of subsystem, possessing no material contacts or other linkages with each other.
2. Establish and solve the equations of each subsystems independently from the other subsystems. One may use any established method of analysis for the subsystems; or one may further subdivide each subsystem and use the the present method to find its solution.
3. Then interconnect the equations of solution of each subsystem by a routine procedure, to arrive at the solution of the original given system.

The first step indicates that the subsystems are appropriately chosen according to the will of the analyst, the storage of the digital computer and the other requirements. They are all independent systems and are connected to the ground at ends of members, which are cut off, to avoid their rigid body movements. If some of them are chosen appropriately to be all the same, their equations are also the same in the second step. This indicates that how to tear apart the system is an important factor to decrease the computation-time. The last step presents the operation to select one of many combinations of solutions of subsystems to construct the original one. In this step, Householder's formula plays an important role.[7] This is a strictly algebraic method of inverting a matrix which is a modification of another matrix whose inverse is already known and is given as follows.

$$(W + XYZ)^{-1} = W^{-1} - W^{-1}X(ZW^{-1}X + Y^{-1})^{-1}ZW^{-1} \qquad (2\text{-}1)^*$$

, where the matrix W, whose inverse is already known, is modified by the addition of the triple product XYZ. The inverse of Y must, of course, exist and the dimensions of X and Z must be suitable; otherwise, X, Y and Z are arbitrary. For the tearing and interconnecting method, W presents a matrix whose diagonal parts are occupied by the stiffness matrices of subsystems, that is, it represents the stiffness matrix of whole structure which consists of a number of independent substructures. Y represents a stiffness matrix of members located between two subsystems before tearing. The matrices X and Z work to enlarge the size of matrix Y to be fit for the size of W and also they work to put the matrix Y at the places where it should be placed. That is, the stiffness matrix Y of a member which located between two joints which are included in different two subsystems is added to those of two joints concerned for the sake of two matrices X and Z.

The matrices which represent the network-topological property of a frame show the

* ; In the case of X = Z in Householder's formula, the equation is called "Bückner's formula".[30]

connectivity relationship between joints and members, therefore the two matrices X and Z can be replaced by the submatrices of those.

2–3. Network-Topological Properties as Information for Tearing and Interconnecting Method

As indicated before, the matrices which show the network-topological properties of the given framed structure present the connectivity relationship of all structural elements which are included in the system. They are called "Branch Node Incidence Matrix" which is noted by A matrix, "Node-to-datum Path Matrix" noted by B_T matrix, "Branch Mesh Matrix" by C matrix and "Basic Cut-set Matrix" by D matrix.[8, 16]

In the network-topological formulation for displacement method, there exist two methods, and one of them is called "Node Method" (ordinary displacement method) in which joint displacement are used as auxiliary variables, and the other is called "Tree Method" in which the distorsions of some members (i.e. tree members) are used as auxiliary variables, and the latter is thought as a variation of the former.[16]

In the displacement method, we must construct the stiffness matrix for whole structure from primitive stiffness matrices which show the rigidity of every structural element (member) of it, and from it the flexibility matrix must be calculated. The tearing of the physical system is identical to that of the matrix which represents its topological property. And also as shown in the previous sentense of the last section, the informations which are necessary for the interconnecting of subsystems can be obtained from the matrix.

In the following, "Node Method" and "Tree Method" are summarized respectively and the network-topological property as information for tearing and interconnecting method, which are applied to displacement method, is also given.

2 3 1. Node Method

In the node method or an ordinary displacement method, the topological property plays the role of "Compatibility condition" and "Equilibrium condition" at each joint. Consequently, the primitive stiffness matrices of all members in the given system, which are calculated as cantilevers whose final ends are fixed, are combined by Branch Node Incidence Matrix, A, to construct the original system. The relation between the applied load vector and the joint displacement vector can be presented as following, if the formulation is performed in the global coordinate system.

$$u = (A^tKA)^{-1} p \qquad (2\ 2)$$

, where u ; Joint displacement vector

p ; Applied joint load vector

K ; Primitive stiffness matrix

(A^tKA) is called "Joint Stiffness Matrix" of the structure and the calculation for the inversion of (A^tKA) is the most troublesome, when the rank of joint stiffness matrix is large enough. Therefore the matrix must be torn into some submatrices in order to overcome the difficulties.

If the given system is divided into N subsystems and their incidence matrices and primitive stiffness matrices are presented as

$$\left.\begin{array}{l} \tilde{A}_1,\ \tilde{A}_2,\ \cdots,\ \tilde{A}_i,\ \cdots,\ \tilde{A}_N \\ \tilde{K}_1,\ \tilde{K}_2,\ \cdots,\ \tilde{K}_i,\ \cdots,\ \tilde{K}_N \end{array}\right| \qquad (2\ 3)$$

, the joint stiffness matrices are respectively given as follows,

$$\tilde{A}_1^t \tilde{K}_1 \tilde{A}_1,\ \tilde{A}_2^t \tilde{K}_2 \tilde{A}_2,\ \cdots,\ \tilde{A}_N^t \tilde{K}_N \tilde{A}_N \qquad (2\ 4)$$

By arranging them appropriately, the joint stiffness matrix of a gathering of subsystems is given as

$$\tilde{A}^t \tilde{K}\tilde{A} = \begin{bmatrix} \tilde{A}_1^t \tilde{K}_1 \tilde{A}_1 & & & 0 \\ & \tilde{A}_2^t \tilde{K}_2 \tilde{A}_2 & & \\ & & \ddots & \\ 0 & & & \tilde{A}_N^t \tilde{K}_N \tilde{A}_N \end{bmatrix} \qquad (2\ 5)$$

This matrix is the one whose diagonal parts are occupied by the submatrices of subsytems and as there are no contacts between these subsystems, they are all independent each other, that is, there are no stiffness elements between two subsystems. Physically, the members which locate between two subsystems are all connected to the datum node (i.e. to the ground) at the ends which are cut off.

The interconnecting of these independent subsystems to construct the original one is equal to filling the stiffness of all members which are cut off at the stage of tearing in the right place of the joint stiffness matrix of torn systems and to find the inverse of it. These treatments can be done by using the branch node incidence matrix of original system only. Rows of incidence matrix indicate the members and every row has plus one and minus one at the places on which the member is incident, and the other elements are all zero. On the contrary, every column matrix of it indicates the joint number and it contains some (+1) and (−1) as many as the members which are connected to the joint.

If the columns are rearranged appropriarely and joints which should be included in the same subsystems are gathered separately, the incidence matrix for original system can be written as

$$A = \left[A_1,\ A_2,\ A_3,\ \cdots,\ A_N \right] \qquad (2\ 6)$$

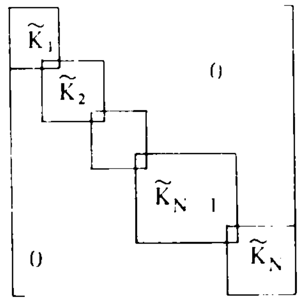
, where the suffix indicates the number of subsystem. And if the elements which are included in the same row are added in the range of submatrix which is indicated by the suffix, some rows are equal to zero and others not equal to zero, but equal to plus one or minus one. Those which are equal to zero mean that both ends of members are included in the same subsystem and those which are not equal to zero mean that the members are located between different two subsystems or they are connected to the datum node and the member of the subsystems where both ends locate is indicated by the number of submatrices where the element, (+1) and (−1), can be found.

If eq. (2-6) is used to construct the joint stiffness matrix, it becomes as follows.

$$A^t KA = \begin{bmatrix} A_1^t KA_1 & A_1^t KA_2 & \cdots & A_1^t KA_N \\ A_2^t KA_1 & A_2^t KA_2 & \cdots & A_2^t KA_N \\ \vdots & \vdots & & \vdots \\ A_N^t KA_1 & A_N^t KA_2 & \cdots & A_N^t KA_N \end{bmatrix} \qquad (2\text{-}7)$$

Now we compare eq. (2-5) with the diagonal parts of eq. (2-7). A_i and $\widetilde{A}_i$ are matrices which contain only the same joints, but $\widetilde{A}_i$ is specified only for the members which are included in the i-th subsystem and A_i is defined for all members in the original system, therefore $\widetilde{A}_i \subset A_i$. The stiffness matrix $\widetilde{K}_i$ is, of course, a submatrix of K and it contains the rigidity coefficients of members which are included in the i-th subsystem. The relation of K and $\widetilde{K}_i$ can be shown as follows.

K =



(2-8)

The elements which are included in two submatrices, $\widetilde{K}_i$ and $\widetilde{K}_{i+1}$, are those that are the members to connect between two subsystems, the i-th and the (i + 1)-th subsystems. The joint stiffness matrix of the i-th subsystem, $\widetilde{A}_i^t \widetilde{K}_i \widetilde{A}_i$, is the one in which only the members in the subsystem are taken into consideration and $A_i^t KA_i$ is the one which is a submatrix of $A^t KA$, in which all members in original system are taken into consideration, and it is defined for joints which are considered in the i-th subsystem. $\widetilde{A}_i$ is thought as a subspace of A_i, and between them we have following relation.

$$A_i^t = \left[\ 0,\ 0,\ \cdots,\ \widetilde{A}_i^t,\ 0,\ \cdots,\ 0\ \right] \qquad (2\text{-}9)$$

Therefore, it concludes that

$$\widetilde{A}_i^t \widetilde{K}_i \widetilde{A}_i = A_i^t KA_i \qquad (2\text{-}10)$$

It means that the joint stiffness matrices of subsystems can be calculated by using original incidence matrix A.

In eq.(2 5), all of the off-diagonal parts of $\tilde{A}^t\tilde{K}\tilde{A}$ are zero, though in eq.(2 7) they are occupied by $A_i^tKA_j$ ($i \neq j$). $A_i^tKA_j$ corresponds to the connectivity relationship between the i-th and the j-th subsystems. Therefore, if they are not directly connected each other, they are equal to zero, otherwise not equal to zero.

Using Householder's formula, (W + XYZ) is equal to the joint stiffness matrix A^tKA and W in eq.(2 1) corresponds to eq.(2 5). XYZ indicates the stiffness matrix of members which connect different two subsystems.

Now we consider the incidence matrix once again. The Branch Node Incidence Matrix A is called "Co-boundary Operator", and its transposed form, A^t, is called "Boundary Operator", and they define the branches which are the co-boundary of nodes and the nodes which are the boundary of branches, respectively. Then, they define the connectivity between the branches which are used for interconnecting and the nodes where the branches are connected. It means that the incidence matrix plays an important role for the stage of interconnecting.

In the case of eq.(2 7), a substructure contains a number of joints and therefore only some members are used as co-boundaries of it. If the number of N in eq. (2 7) is equal to the number of joints, every suffix indicates the column number. It means that the given system is divided into the number of joints and every subsystem consists of only one joint and some members which are connected to the joint. Every column of the incidence matrix contains some (+1) and (1). The row number of the column matrix where (+1) or (1) locates presents the number of member which is connected to the joint. Zero element of the column matrix indicates that the member where zero exists is not incident on the joint. In this tearing method, all the members of the original system become the co-boundaries. Thus, all of them are used to connect the subsystems in the stage of interconnecting.

This tearing method is based on the concept of ultimate building-block being one-joint structure. On the other hand, we can consider a member being an ultimate building-block. Both of them are presented in Section 2 4 more precisely.

2 3 2. Tree Method

Tree method is also a kind of displacement method and is thought as a variation of the node method, because it can be derived from the node method applying another network-topological property, i.e. Node-to-datum path matrix B_T. But this method has an important property which is quite different from the node method. That is, though the node method treats the given system as it be (namely, it treats the framed system as indeterminate one), all the structures being analyzed are transformed into equivalent determinate systems in this method Ordinary framed systems have a number of closed pathes. If a member, which is selected arbitrarily within the members which construct a closed path, is cut off, it becomes a determinate one, but it behaves different from the one of original system. To insure the identical behaviour, the equivalent stiffness of the cut-off-members must be added to the rest member. In tree method, this operation is automatically done and it treats with transformed equivalent determinate systems only.

The basic equation to calculate the member distorsion induced by the nodal loads is presented as following, when it is formulated in the global coordinate system.

$$v_T = (D^t KD)^{-1} B_T p \qquad (2\text{-}11)$$

, where v_T ; Induced distorsion vector of tree members

p ; Applied joint load vector

D ; Basic cut-set matrix

B_T ; Node-to-datum path matrix

D^tKD is "Joint stiffness matrix". As two matrices, D and K, can be divided into two parts, i.e. tree and link parts, the joint stiffness matrix can be treated as

$$D^tKD = \begin{bmatrix} I_T, & D_L^t \end{bmatrix} \begin{bmatrix} K_T & 0 \\ 0 & K_L \end{bmatrix} \begin{bmatrix} I_T \\ D_L \end{bmatrix} = K_T + D_L^t K_L D_L \qquad (2\text{-}12)$$

, where I is a unit matrix and the subscripts, T and L, indicate "tree" and "link". Eq. (2-12) shows the most important property of this method, that is, the stiffness of link members is transformed into equivalent stiffness which corrects the stiffness of adjacent tree members and the given system is transformed into equivalent determinate system.

In the calculation of eq. (2-12), the inversion of (D^tKD) is the most troublesome. At this stage, piecewise analysis is applied. And eq. (2-12) becomes more important, because it presents not only the decreasing of calculation-time to obtain the joint stiffness, but also the suggestion for tearing and interconnecting method in Tree system. The first term of right side of eq. (2-12) shows the stiffness matrix of tree members only. As the tree system is, of course, an determinate system, its inverse can be obtained easily. The second term of it indicates the method for interconnecting by which the separated independent tree members are connected each other.

At this stage, let us examine the property of the basic cut-set matrix, D. When the tree system of the given structure is selected arbitrarily, the number of tree members is always equal to that of joints. If the original system is arbitrarily divided into two parts, that is, one of them includes the datum node and the other does not include it, we find a lot of cut-lines which partition the system into two parts. But, if a cut-line is restricted as to cut off only one tree member, the number of the cut-off lines is equal to that of tree members. Basic cut-set matrix shows which members are cut off by each cut-line and those which are cut off are indicated by (+1) or (-1), and those which are not cut off are indicated by (0). The signs (+) and ( ) are determined whether the orientation of the member concerned is identical to the one of the tree member which is just now cut off. Therefore, $D_T = I_T$ in eq.(2 12).

As recognized from eq. (2 12), the rigidities of link members are distributed not only to the tree members which are directly connected to them but also to some of those tree members which are located between the trees being directly connected and the datum node. This is caused by D matrix, and more precisely by B_T matrix. The rigidity of a link member is, thus, added to those of tree members whose basic cut-sets include the link member. This relation between link members and tree members can be shown by row matrices of D_L.

As well as the row matrix of D_L, the column matrices of D also present the relation between them. And the i-th column matrix presents which members construct the boundary between two adjacent sub-systems.

Now we consider the stage of interconnecting. A given system can be divided into an arbitrary number of sub-systems, but they should be chosen as follows; there was only one tree member between two adjacent sub-systems before tearing. This means that the division is presented as one of the basic cut-set of the given system. Therefore, the equation (2 12) of the original system can be used in this stage and only the basic cut-set matrix of link members becomes important.

For the interconnecting of two adjacent subsystems, there are two typical methods, that is, in one method we pay attention to the basic cut-set and in the other method we pay attention to a link member. Namely we have the following methods;

(1) By use of the column matrices of the basic cut-set matrix, two subsystems are combined by adding the rigidities of all link members which are incident on both basic cut-sets of two subsystems. That is, if D_i and D_j are two column matrices which are selected from two subsystems, $D_i^t K D_j$ indicates the rigidities of link members which are incident on both of the i-th and the j-th basic cut-set, and it locates at (i, j)-th element of original joint stiffness matrix, $D^t K D$.

(2) By use of the row matrices of link part of the basic cut-set matrix, every link member is connected between two subsystems step by step. That is, paying attention to one link member, the rigidities of tree members concerned are modified by adding that of the link member. And this operation is done for all link members which locate between two subsystems before tearing.

These two typical methods are based on the basic cut-set matrix. As far as the network-topological property is used as informations for tearing and interconnecting, they are basic methods for all piecewise analysis.

In Section 2 5, we present the piecewise analysis for the tree method according to these two typical methods. And, as well as Section 2 4, they are illustrated in accordance with the concept of "ultimate building-block" of the tree method.

2–4. Application of Piecewise Analysis to Node Method

A framed structure can be thought as a gathering of a number of members and their joints. When it is torn ultimately, two kinds of ultimate building-block can be thought;

1. The system is torn as many as the number of joints. A subsystem consists of only one joint and some members whose one end is connected to the joint and the other end is connented to the datum node. (See Fig.2–1)
2. The system is torn as many as the number of members. Therefore, a subsystem consists of only one member which is thought as a cantilever.

These ultimate building-blocks are connected one after another till they form the original system and at the same time the flexibility of the given system is obtained by successive application of Householder's formula.

2–4–1. In the case of tearing a system as many subsystems as the number of joints.[16]

When the given system is as shown in Fig. 2–1 a, the other ends of all members, whose one end is connected to a joint, are cut off and are connected to the datum node (i.e. to the ground). This operation is repeated for every joint of the system and the original one is replaced by a gathering of as many subsystems as the number of joints, as shown in Fig.2–1 b. In this case, it consists of three sybsystems.

If the original system has n joints, it is torn into n subsystems. The branch node incidence matrix, A, can be divided into n column matrices and it is given as follows,

$$A = \left[A_1,\ A_2,\ \cdots,\ A_i,\ \cdots,\ A_n \right] \tag{2–13}$$

, where A_i is the i-th column matrix of the incidence matrix and its subscript, i, indicates the number of the joint, i.e. the i-th joint of the system.

Some elements of the column matrix A_i are (+1) and (–1) and others are all zero. The row number whose element is occupied by either (+1) or (–1) indicates the member being connected to the i-th joint.

The joint stiffness matrix of the system is given as

$$A^t K A \tag{2–14}$$

In this equation, K indicates the primitive stiffness matrix of the members and it is expressed as follows,

$$K = \left[\diagdown K_i \right] \tag{2–15}$$

, where K_i is a stiffness matrix of the i-th member and it is calculated by assuming the i-th member fixed at its final end, in other words, as a cantilever.

Using eq.(2-13) in eq.(2-14), the joint stiffness matrix A^tKA can be written as

$$A^tKA = \begin{bmatrix} A_1^tKA_1, & A_1^tKA_2, & \cdots & A_1^tKA_n \\ \vdots & \vdots & & \vdots \\ A_i^tKA_1, & A_i^tKA_2, & \cdots & A_i^tKA_n \\ \vdots & \vdots & & \vdots \\ A_n^tKA_1, & A_n^tKA_2, & \cdots & A_n^tKA_n \end{bmatrix} \qquad (2\text{-}16)$$

In this equation, the off-diagonal element, $A_i^tKA_j$, indicates the influence of the stiffness of a member, which locates between the i-th and the j-th joints, to the joint stiffness matrix. If the stiffness of the member is noted by K_{ij}, the relation between $A_i^tKA_j$ and K_{ij} is

$$A_i^tKA_j = K_{ij} \qquad (2\text{-}17)$$

, because the initial and the final ends of a member are indicated by (+1) and (−1), respectively, and one of the k-th rows of A_i and A_j is occupied by (+1) and the other by (−1), if the member is the k-th member.

The diagonal element, $A_i^tKA_i$, presents a joint stiffness of the i-th joint where a number of members are connected. And it is equivalent to that of a system which has only one joint and some members, one ends of which are connected to the joint and the others to the datum node as shown in Fig. 2-1 b. That is, all diagonal elements present stiffness coefficients of one-joint structures which are independent each other.

Then, if the stiffness matrix of torn system is noted by $K^{(0)}$, it is defined as follows,

$$K^{(0)} = \begin{bmatrix} A_1^tKA_1 & & & & 0 \\ & A_2^tKA_2 & & & \\ & & \ddots\ A_i^tKA_i & & \\ & 0 & & \ddots & A_n^tKA_n \end{bmatrix} \qquad (2\text{-}18)$$

A permutation matrix Ω_i is defined as follows,

$$\Omega_i = \left[0,\ 0,\ \cdots,\ I_i,\ 0,\ \cdots,\ 0\right] \qquad (2\text{-}19)$$

, where 0 ; A (6 X 6) zero matrix for space system and a (3 X 3) zero matrix for plane one

I_i ; Unit matrix, which locates at the i-th column in expression for the permutation matrix

Using eqs.(2-17),(2-18) and (2-19), the joint stiffness matrix of the given system yields to

$$A^tKA = K^{(0)} + \sum_{i,j=1}^{n} \Omega_i^t\, K_{ij}\, \Omega_j \tag{2-20}$$

, where K_{ij} is the primitive stiffness matrix of a member which is connected to the i-th and the j-th joint, namely $K_{ii}=0$.

The first term of right side of eq.(2-20) presents the stiffness matrix of a system which consists of n independent one-joint structures, and original stiffness can be obtained by successive addition of the second term to it and this operation means the method of interconnecting.

For the first step to calculate the joint flexibility matrix, F, of the given system from eq.(2-20), the flexibility matrix $F^{(0)}$ of a system which consists of n independent one-joint structures is defined as follows,

$$F^{(0)} = \left[K^{(0)} \right]^{-1} \tag{2-21}$$

$F^{(0)}$ can be evaluated easily, because $K^{(0)}$ has the form as shown in eq.(2-18) and $A_i^tKA_i$ is a (3 X 3) nonzero matrix for a plane structure and a (6 X 6) matrix for space one.

Now, Householder's formula is successively applied to eq.(2-10) to obtain $(A^tKA)^{-1}$, i.e. the joint flexibility matrix. By connecting the i-th and the j-th joints by a member whose stiffness is noted by K_{ij}, the joint stiffness matrix of torn system is added by ($\Omega_i^tK_{ij}\Omega_j + \Omega_j^tK_{ji}\Omega_i$). This operation is done for all members. And at every addition of a member, Householder's formula is applied.

The inverse matrix $F^{(1)}$ of the sum of the first term and one symmetric element, ($\Omega_i^t K_{ij}\, \Omega_j + \Omega_j^t K_{ji}\, \Omega_i$), of the second term of the right side of eq.(2-20) can be calculated by the application of eq.(2-1) and the kl-th element of $F^{(1)}$ is obtained as

$$\begin{aligned} F_{kl}^{(1)} = {} & F_{kl}^{(0)} + F_{ki}^{(0)} K_{ij} F_{jl}^{(0)} + (F_{kj}^{(0)} + F_{ki}^{(0)} K_{ij} F_{jj}^{(0)}) \\ & \times (K_{ij}^{-1} - F_{ii}^{(0)} K_{ij} F_{jj}^{(0)})^{-1} (F_{il}^{(0)} + F_{ii}^{(0)} K_{ij} F_{jl}^{(0)}) \end{aligned} \tag{2-22}$$

, where $F_{kl}^{(0)}$ is the kl-th element of a matrix $F^{(0)}$. $F^{(1)}$ means the flexibility matrix of a system whose i-th and j-th joints are connected each other and the other joints are still independent one-joint structures.

By successive application of Householder's formula for all i and j, the kl-th element of the joint flexibility matrix F of the original system is found as

$$\begin{aligned} F_{kl} = F_{kl}^{(m)} = {} & F_{kl}^{(m-1)} + F_{ki}^{(m-1)} Z_{ij}^{(m-1)} F_{jl}^{(m-1)} + (F_{kj}^{(m-1)} + F_{ki}^{(m-1)} Z_{ij}^{(m-1)} F_{jj}^{(m-1)}) \\ & \times (Z_{ij}^{(m-1)} - F_{ii}^{(m-1)} Z_{ij}^{(m-1)} F_{jj}^{(m-1)})^{-1} (F_{il}^{(m-1)} + F_{ii}^{(m-1)} Z_{ij}^{(m-1)} F_{jl}^{(m-1)}) \end{aligned} \tag{2-23}$$

, and $m = 1, 2, \cdots, M$.

$F_{ij}^{(m-1)}$; the (i, j)th element of matrix $F^{(m-1)}$ which is calculated by successive application of (m 1) times Householder's formula

M ; the number of members which are not connected with the support joints

$Z_{ij}^{(m-1)} = (K_{ij}^{(m)} \quad F_{ji}^{(m-1)})^{-1}$

In Figs. 2 1 c and 2 1 d, the divided substructures are connected one after another by two members whose stiffness coefficients are indicated by K_{12} and K_{23}, respectively. In this example, there are two members which are not connected to the support joints, so M = 2.

At every intermediate step for the calculation of F of given structure, we need not calculate the flexibility matrix of intermediate step with respect to whole k and l but may calculate only some elements of the matrix which are modified by the connection of a member at the step.

2 4–2. In the case of tearing a system as many subsystems as the number of members

All the members in a framed structural system are arbitrarily classified into two kinds, i.e. tree and link members. By the proper rearrangement of row matrices of the branch node incidence matrix of the given system, it can be written as follows,

$$A = \begin{bmatrix} A_T \\ A_L \end{bmatrix} \tag{2 24}$$

, where A_T and A_L present the submatrices showing the connectivity relationships only for tree and link members, respectively.

According to this rearrangement, the primitive stiffness matrix K is also written as,

$$K = \begin{bmatrix} K_T & 0 \\ 0 & K_L \end{bmatrix} \tag{2–25}$$

Using these two equations, the joint stiffness matrix of the system is given as follows,

$$A^t K A = A_T^t K_T A_T + A_L^t K_L A_L \tag{2 26}$$

If the given system contains m link members, A_L and K_L can be given as follows,

$$A_L = \begin{bmatrix} A_L^1 \\ A_L^2 \\ \cdot \\ \cdot \\ A_L^m \end{bmatrix}, \qquad K_L = \begin{bmatrix} K_L^{11} & & & 0 \\ & K_L^{22} & & \\ & & \ddots & \\ 0 & & & K_L^{mm} \end{bmatrix}$$

Using these two relations, the second term of the right side of eq. (2-26) becomes as following,

$$\begin{aligned} A_L^t K_L A_L &= A_L^{1t} K_L^{11} A_L^1 + A_L^{2t} K_L^{22} A_L^2 + \cdots + A_L^{mt} K_L^{mm} A_L^m \\ &= \sum_{i=1}^{m} A_L^{it} K_L^{ii} A_L^i \end{aligned} \tag{2-27}$$

Then, the joint stiffness matrix is given as

$$A^t KA = A_T^t K_T A_T + \sum_{i=1}^{m} A_L^{it} K_L^{ii} A_L^i \tag{2-28}$$

The first term of the right side of this equation presents the joint stiffness matrix of a tree system which is, of course, a determinate system and by adding a term, $A_L^{it} K_L^{ii} A_L^i$, to it, the determinate system becomes an indeterminate one and at the last stage,i.e. $i = m$, it gives the original stiffness matrix. That is, this equation suggests the method for tearing and interconnecting. At first, all the members of the system are torn apart each other and they are fixed at their final joints. In other words, the given system is torn into a system which consists of cantilevers. Thus as many number of members as that of joints are selected as tree members. Then, the joint stiffness matrix of only tree members can be calculated.

In the stage of interconnecting, Householder's formula is applied to eq. (2-28) in order to find its flexibility. And $(A_T^t K_T A_T)^{-1}$ must be calculated, but we need not have the inversion. Between the branch node incidence matrix and the node-to-datum path matrix, we have following relation.

$$A_T^{-1} = B_T^t \tag{2-19}$$

Then, the joint flexibility of tree system is obtained.

$$(A_T^t K_T A_T)^{-1} = B_T^t F_T^{(0)} B_T \tag{2-30}$$

, where $K_T^{-1} = F_T^{(0)}$. Here, we note the joint stiffness and flexibility matrices by $K^{(0)}$ and $F^{(0)}$, respectively.

$$K^{(0)} = A_T^t K_T A_T \tag{2-31}$$

$$F^{(0)} = B_T^t \, F_T^{(0)} \, B_T \tag{2-32}$$

By adding a link member, its stiffness $K^{(1)}$ can be obtained as

$$K^{(1)} = K^{(0)} + A_L^{1t} \, K_L^{11} \, A_L^{1} \tag{2-33}$$

Then, the flexibility matrix becomes as follows,

$$\begin{aligned} F^{(1)} = \left[K^{(1)} \right]^{-1} = B_T^t \Big[F_T^{(0)} - F_T^{(0)} B_T A_L^{1t} \\ \times (F_L^{11} + A_L^{1} B_T^t F_T^{(0)} B_T A_L^{1t})^{-1} A_L^{1} B_T^t F_T^{(0)} \Big] B_T \end{aligned} \tag{2-34}$$

The branch node incidence matrix, the node-to-datum path matrix and the basic cut-set matrix are related as

$$A_L \, B_T^t = D_L \tag{2-35}$$

, so that

$$A_L^i \, B_T^t = D_L^i$$

, where the superscript, i, means its row number.

Using this equation, eq. (2-34) becomes as follows.

$$F^{(1)} = B_T^t \, F_T^{(1)} \, B_T \tag{2-36}$$

, where $$F_T^{(1)} = F_T^{(0)} - F_T^{(0)} D_L^{1t} (F_L^{11} + D_L^{1} F_T^{(0)} D_L^{1t})^{-1} D_L^{1} F_T^{(0)} \tag{2-37}$$

At the m-th addition of a member, the system becomes its original one. Then, the joint stiffness and the flexibility matrices are given as follows,

$$A^t K A = K^{(m)} = K^{(m-1)} + A_L^{mt} \, K_L{}^{mm} \, A_L^{m} \tag{2-38}$$

$$(A^t K A)^{-1} = F^{(m)} = \left[K^{(m)} \right]^{-1} = B_T^t \, F_T^{(m)} \, B_T \tag{2-39}$$

, where $$F_T^{(m)} = F_T^{(m-1)} - F_T^{(m-1)} D_L^{mt} (F_L^{mm} + D_L^{m} F_T^{(m-1)} D_L^{mt})^{-1} D_L^{m} F_T^{(m-1)} \tag{2-40}$$

This method is visualized in an example in Fig. 2-2. In this example, members, 1, 2, 3, and 4, are selected as tree members, while 5 and 6 are link members.

In this method, $F_T^{(i)}$ is successively calculated from i=0 to i=m, and B_T^t and B_T are multiplied to $F_T^{(m)}$ from left and right side, respectively, only for once.

2-5. Application of Piecewise Analysis to Tree Method

When a given system has n joints except its datum node, only n members are classified to be trees and others are all link members. Therefore, there exists n basic cut-sets.

Followingly, the author presents two typical methods of piecewise analysis for tree method which correspond to those described in Section 2 3 2.

2 5 1. In the case of tearing a system into as many subsystems as the number of joints

The basic cut-set matrix of the given system has the order of (number of members X number of joints). Taking notice of the column matrices of it, it can be written as following,

$$D = \left[D_1, D_2, \cdots, D_i, \cdots, D_n \right] \tag{2-41}$$

, where D_i indicates the i-th column matrix of D.

The joint stiffness matrix of the system becomes as following.

$$D^t KD = \begin{bmatrix} D_1^t KD_1 & \cdots & D_1^t KD_i & \cdots & D_1^t KD_n \\ \vdots & & \vdots & & \vdots \\ D_i^t KD_1 & \cdots & D_i^t KD_i & \cdots & D_i^t KD_n \\ \vdots & & \vdots & & \vdots \\ D_n^t KD_1 & \cdots & D_n^t KD_i & \cdots & D_n^t KD_n \end{bmatrix} \tag{2 42}$$

, in which $D_i^t KD_j$ is the sum of stiffness of members which are included in both of the i-th and the j-th basic cut-sets. As described in Section 2 3-2, the basic cut-set matrix can be divided into two parts, i.e. tree and link parts. Therefore, the i-th column matrix is written as

$$D_i = \begin{bmatrix} D_T^i \\ D_L^i \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ I_i \\ 0 \\ \vdots \\ D_L^i \end{bmatrix} \tag{2-43}$$

, where I_i indicates a unit matrix which locates at the i-th row. Then, $D_i^t KD_j$ can be calculated as follows,

$$\begin{cases} D_i^t KD_j = K_T^i + D_L^{it} K_L D_L^i , & \text{if } i=j \\ D_i^t KD_j = D_L^{it} K_L D_L^j , & \text{if } i \neq j \end{cases} \tag{2 44}$$

That is, $D_i^t KD_i$ means a submatrix of diagonal part of D^tKD and it is the sum of stiffness matrices of members which consist of the i-th tree member and some link members that are included in the i-th basic cut-set. And $D_i^tKD_j$ is a stiffness matrix which consists of those of link members.

Now, we cut off all the linkages between tree members in order to construct as many independent subsystems as the number of tree members. That is, as a system which consists of only tree members is an determinate system, all link members of original system are cut off and their rigidities are respectively added to those of tree members on which they are incident. Therefore, the joint stiffness matrix of this transformed system is given as

$$K^{(0)} = \begin{bmatrix} D_1^tKD_1 & & & & 0 \\ & \ddots & & & \\ & & D_i^tKD_i & & \\ & & & \ddots & \\ 0 & & & & D_n^tKD_n \end{bmatrix} \tag{2–45}$$

, in which $D_i^tKD_i = K_T^i + D_L^{it}K_LD_L^i$.

The sum of stiffness of link members which are included in both of the i-th and the j-th basic cut-sets is noted by K_{ij}.

$$K_{ij} = D_L^{it}K_LD_L^j \tag{2–46}$$

Now, we introduce a permutation matrix Ω_i.

$$\Omega_i = \begin{bmatrix} 0, & 0, & \cdots, & I_i, & 0, & \cdots, & 0 \end{bmatrix} \tag{2–47}$$

Ω_i is a row matrix and its i-th column is occupied by a unit matrix and the others all zero matrices.

Using this permutation matrix, the joint stiffness matrix of the original system is written as follows,

$$D^tKD = K^{(0)} + \sum_{i,j=1}^{n} \Omega_i^t K_{ij} \Omega_j \quad (i \neq j) \tag{2–48}$$

This equation indicates the method for interconnecting. That is, by adding the stiffness K_{ij} to $K^{(0)}$, the joint stiffness matrix $K^{(0)}$ of the determinate system is modified. And by the application of Householder's formula to it, the flexibility matrix can be easily calculated. This operation is applied to all i and j.

Note that eq. (2–48) is the similar form of eq. (2–20). Therefore, the method in Section 2–4–1 is applicable in this case.

The flexibility matrix of the transformed tree system is noted by $F^{(0)}$.

$$F^{(0)} = \left[K^{(0)} \right]^{-1} \tag{2-49}$$

The inverse matrix $F^{(1)}$ of the sum of the first term and one symmetric element, $(\Omega_i^t K_{ij} \Omega_j + \Omega_j^t K_{ji} \Omega_i)$, of the second term of the right side of eq. (5–8) can be calculated by the application of eq. (2–1) and the kl-th element of $F^{(1)}$ is obtained as follows,

$$\begin{aligned} F_{kl}^{(1)} = {} & F_{kl}^{(0)} - F_{ki}^{(0)} K_{ij} F_{jl}^{(0)} - (F_{kj}^{(0)} - F_{ki}^{(0)} K_{ij} F_{jj}^{(0)}) \\ & \times (K_{ij}^{-1} - F_{ii}^{(0)} K_{ij} F_{jj}^{(0)})^{-1} (F_{il}^{(0)} - F_{ii}^{(0)} K_{ij} F_{jl}^{(0)}) \end{aligned} \tag{2-50}$$

, where $K_{ij} = D_L^{it} K_L D_L^j$

By successive application of Householder's formula for all i and j, the kl-th element of the joint flexibility matrix of the given system is found as follows,

$$\begin{aligned} F_{kl} = F_{kl}^{(n)} = {} & F_{kl}^{(n-1)} - F_{ki}^{(n-1)} Y_{ij}^{(n)} F_{jl}^{(n-1)} - (F_{kj}^{(n-1)} - \\ & F_{ki}^{(n-1)} Y_{ij}^{(n)} F_{jj}^{(n-1)}) (Y_{ij}^{(n)-1} - F_{ii}^{(n-1)} Y_{ij}^{(n)} F_{jj}^{(n-1)})^{-1} \\ & \times (F_{il}^{(n-1)} - F_{ii}^{(n-1)} Y_{ij}^{(n)} F_{jl}^{(n-1)}) \end{aligned} \tag{2-51}$$

in which

$F_{ij}^{(n-1)}$; the (i, j)th element of matrix $F^{(n-1)}$ which is calculated by $2(n-1)$ times successive applications of Householder's formula

n ; the number of basic cut-set, and

$$Y_{ij}^{(n)} = \left[K_{ij}^{(n)-1} + F_{ji}^{(n-1)} \right]^{-1}$$

An example of this method is illustrated in Fig. 2–3.

Fig. 2–3–a shows the given system and real lines and dotted lines indicate tree and link members, respectively. Figs. 2–3–b and 2–3–c show the step of adding the rigidities, K_{12} and K_{23}, respectively. In this example, K_{12} and K_{23} are equal to the rigidities of members which are indicated by 4 and 5. And K_{13} is equal to zero in this case, therefore we need only two steps to obtain the joint flexibility matrix of the given system.

2–5–2. In the case of tearing a system into as many subsystems as the number of members

This method is the one in which every column matrix of the basic cut-set matrix of the given system is used as an information for interconnecting.

As an ultimate building-block of the given framed structure, let us take a tree member whose final end is fixed to the datum node and whose initial end is free. By adding one link member to the system consisting of only tree members, the stiffness of a number of joints which are concerned to the link member is modified and the flexibility matrix of the system with all the tree members and only one link member is obtained by the application of Householder's formula. This operation is repeated as many as the number of link members, till the joint flexibility matrix of the original system is obtained.

The joint stiffness matrix of the given system is presented as shown in eq. (2–12).

$$D^t K D = K_T + D_L^t K_L D_L \qquad (2\text{–}12)$$

Taking notice of the row matrices of D, the basic cut-set matrix can be written as follows,

$$D^t = \left[I_T,\ D_L^{1t},\ D_L^{2t},\ \cdots,\ D_L^{it},\ \cdots,\ D_L^{mt} \right] \qquad (2\text{–}52)$$

, where the superscript i indicates the number of link member.

Using eq. (2–52), eq. (2–12) becomes as follows,

$$D^t K D = K_T + \sum_{i,j=1}^{m} D_L^{it} K_L^{ij} D_L^{j} \qquad (2\text{–}53)$$

If $i \neq j$, $D_L^{it} K_L^{ij} D_L^{j} = 0$, because K_L matrix is a diagonal matrix. Then, eq. (2–53) yields to be

$$D^t K D = K_T + \sum_{i=1}^{m} D_L^{it} K_L^{ii} D_L^{i} \qquad (2\text{–}54)$$

This equation indicates the interconnecting method of successive addition of link members to the system consisting of only tree members.

By adding the first link member to the tree system, the joint flexibility matrix $F^{(1)}$ is obtained as follows,

$$\begin{aligned} F^{(1)} &= \left[K^{(1)}\right]^{-1} = \left[K_T + D_L^{1t} K_L^{11} D_L^{1}\right]^{-1} \\ &= F_T - F_T D_L^{1t}\left[F_L^{11} + D_L^{1} F_T D_L^{1t}\right]^{-1} D_L^{1} F_T \end{aligned} \qquad (2\text{–}55)$$

, where $F_T = \left[K_T\right]^{-1}$, $F_L^{11} = \left[K_L^{11}\right]^{-1}$

By the m-times applications of Householder's formula to eq. (5–14), the joint flexibility matrix of the given system is obtained.

$$\begin{aligned} F = F^{(m)} &= F^{(m-1)} - F^{(m-1)} D_L^{mt} \left[F_L^{mm} \right.\\ &\quad \left. + D_L^{m} F^{(m-1)} D_L^{mt}\right]^{-1} D_L^{m} F^{(m-1)} \end{aligned} \qquad (2\text{–}56)$$

, where $F^{(m-1)}$ presents the joint flexibility of the system to which the (m–1)–th link member is connected.

Using this method, every link member is connected to the tree system one after another, then we must repeat the operation as many times as the number of link members.

2–6. Rearrangement of Joints for Piecewise Analysis

In order to use general piecewise analysis effectively, one must rearrange the ordering of joints before the application of piecewise technique. But, if the proposed piecewise technique in previous sections is used, the rearrangement of joints is unecessary, because a structural system is divided into as many subsystems as the number of joints or members, and the stiffness matrix of a subsystem contains the least number of zero elements. On the other hand, as far as a system is to be divided into arbitrary number of subsystems, the procedure for the rearrangement of joints is needed.

Arbitrary labeling of joints spreads non-zero elements in the stiffness matrix, though the matrix has the characteristic of gathering non-zeros within any width along principal diagonal by proper labeling.

If a system is divided into two parts, i.e. I and II subsystems, joints are also classified into two parts. The classification of joints without using original figure is very difficult, though the stiffness matrix contains the information. If the graphical figure is used, the work is easily proceeded. For the operation of classification of joints, actual system is not needed but the graph which shows the connectivity relationship of structural elements. Furthermore, as far as the piecewise method is used, the classification of joints is the division of the group of joints into as many groups as the number of subsystems. The rearrangement of joints in a subsystem is not needed for the method. That is, even if non-zero elements in a subsystem may have the tendency to gather within a narrow bandwidth, it is not needed to gather them for the method. Only the necessity for the piecewise method is the classification and rearrangement of joints as many groups as the number of subsystems. That is, Kron's piecewise method leaves the useful property of stiffness matrix unused. Thus, if the band matrix method is used for the actual calculation of subsystems and the Householder's formula is used for the interconnecting of the results, more saving of memory and calculation-time can be expected. But for the band matrix method the rearrangement of joints within a subsystem has also to be proceeded.

The tearing of a system is operated in accordance with the graph of a system and the classification of joints is very easy, but the rearrangement of joints for the band matrix method is very difficult, even if the graph is used.

2–7. Conclusions

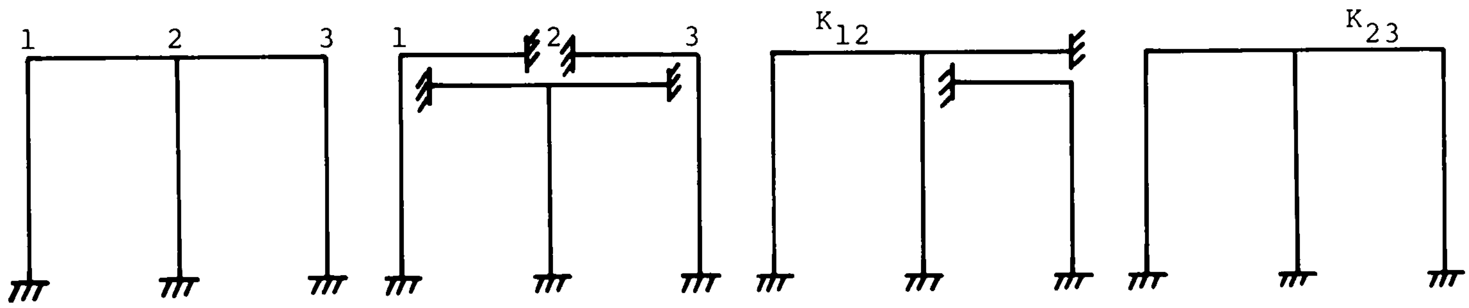
In the investigations of this chapter, the piecewise method which is one of the efficient methods for the saving of memory and calcualtion-time is introduced.

Using the method, the network-topological property of structural system is usefully introduced and it gives the informations for tearing the system.

In accordance with the informations the author proposed a kind of modifications of Kron's piecewise analysis, that is, tearing a system as many substructures as the number of structural elements.

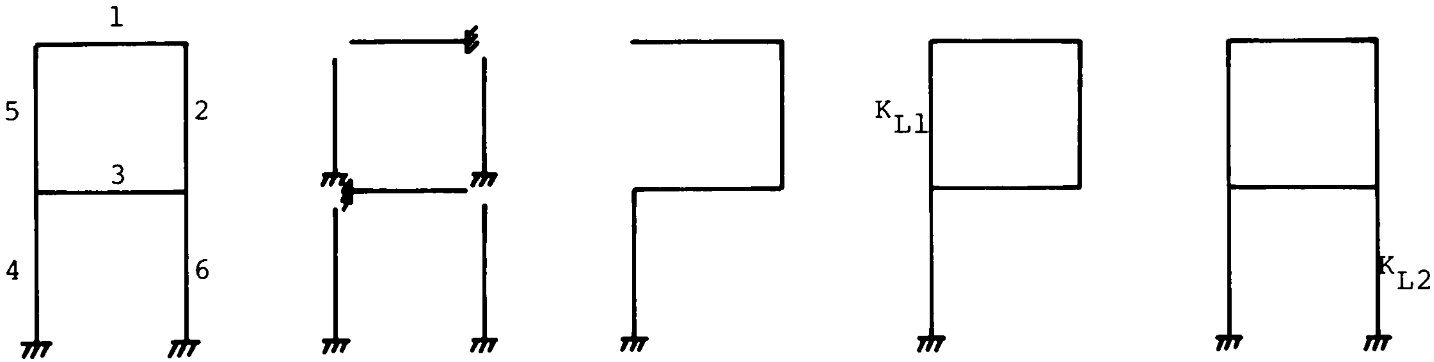
In general piecewise application, the rearrangement of joints is needed but for the application of the proposed method, the procedure is useless.

Generally speaking, Kron's method for piecewise analysis cann't use effectively the property of structural stiffness matrix for a huge structure. Thus, in order to use the property more effectively the band matrix method should take place in the stage of calculation of each subsystem and the Householder's formula may be used to combine the results obtained for subsystems. But, the band matrix method can play the role most effectively, if the minimum bandwidth of a stiffness matrix is obtained.



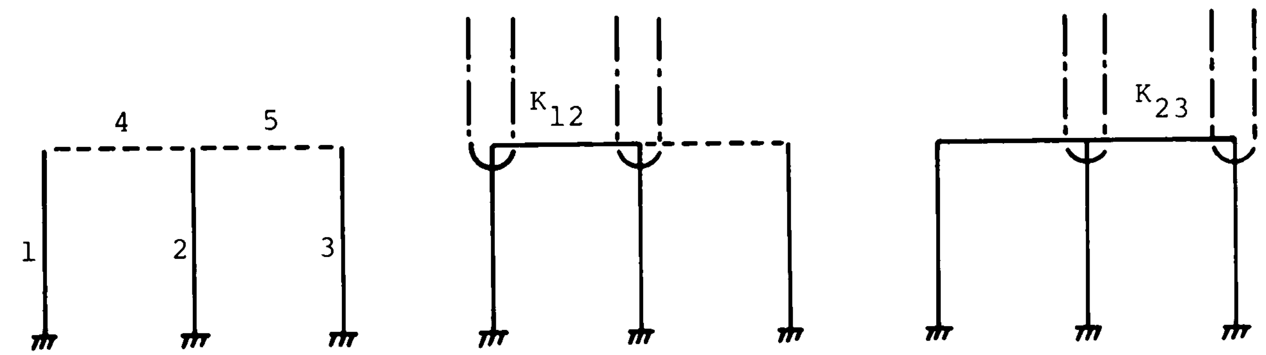
(a) Given System (b) Torn System (c) $F^{(1)}$ (d) $F^{(2)} = F$

Fig. 2-1 Piecewise Analysis of Node Method (I)



(a) Given System　(b) Torn System　(c) Tree System　(d) $F^{(1)}$　(e) $F^{(2)} = F$

Fig. 2–2 Piecewise Analysis of Node Method (II)



(a) Given System

(b) $F^{(1)}$

(c) $F = F^{(2)}$

Fig. 2-3 Piecewise Analysis of Tree Method

CHAPTER 3
TOPOLOGY OF STRUCTURE AND BANDWIDTH PROBLEM

3-1. Introduction

Structural systems in the field of civil engineering are composed of many types of structural elements with distinguished properties. These elements are placed and connected each other in a structure so as their properties can be used most effectively.

Let consider a suspension bridge, for example, the Severn Bridge in Scottland. In the structure we can find out cable structures for main cables and hangers, member structures for towers and plate structures for stiffening box girders. For the structural element which should resist only the tensile force, cable structure is the best. Against the torsion and the bending force, box type structure can resist well among various types of structural systems with the same weight. And, to carry the vertical loads from the main cables column structure may be the best one.

Any civil engineering structure has, of course, a three-dimensional configuration and the structural elements are also the three-dimensionals. But their main mechanical properties which are used in the total system may not be considered to be same as the dimensions of their configurations, but equal to or less than them.

Take a cable structure as an example. It can, of course, resist the bending force, but it is disregarded and only the resistance along its longitudinal axis is meaningful. In the sense, the element may be considered as a one-dimensional structural system.

Generally, structural elements are classified as followings.

(1) One-dimensional elements

(2) Two-dimensional elements

(3) Three-dimensional elements

Their boundaries are nodes, lines, and surfaces, respectively. On these boundaries the elements are connected each other.

On the other hand, the development of the digital computers increased the importance of the matrix method for actual structural analysis in which any structural system is treated as a gathering of small structural elements with simple boundaries and the behaviour of whole system is measured at the boundaries of the elements. Furthermore, for the convienience of the treatment of boundary, some representative nodes on them in stead of line and surface boundries are chosen, i.e. the former by the both ends and the latter by the edge nodes of the surfaces. That is, any structure is replaced by a gathering of nodes which are selected in order to express the behaviour of original structure.

These treatments are only for the convienience of actual calculation and aren't done by the equivalence of an actual structure and its mathematical model. Therefore, we need, at least, the insurance that the solution of the convienient form can sufficiently present the behaviour of the original one.

A mathematical system with nodes and lines is called a linear graph. That is, the model which is expressed by matrix method is equivalent to a graph. As described in Chapter 1, one of the efficient method for the inversion of a matrix is called "Band Matrix Method". In order to draw out the merit of the application of the method, the bandwidth of a matrix which is to be inverted should be reduced as small as possible. The bandwidth is decided by the nodal-labeling of a graph corresponding to a structural system. That is, how to label nodes of a graph decides the bandwidth of the matrix.

In this chapter, graph theory is, at first, introduced and some definitions of the theory are explained. They can help the understanding of following chapters. Followingly, graphical representation of structures is explained and using these results, topological considerations on bandwidth problem are done.

3-2. Graph Theory

As described in previous section, the behaviour of any structural system is measured at a number of nodes on it and the mechanical property must be transmitted along lines which connect the representative nodes. Therefore, any structure can be replaced by a system which consists of nodes and lines.

The new system with nodes and lines constructs a network in which physical properties in original system are approximately taken into consideration.

By removing all of the physical properties from the transformed network system, there leave only nodes and lines. It is called "a graph". As far as values of a stiffness matrix of an actual structural system are our object, physical properties are the most important. But, if the connectivity relationship of nodes in the system is our concern, all of the physical properties of nodes and lines become meaningless and they should be removed from them for the simplicity of the problem.

Graph theory treats the graph and clarifies the characteristics of it. Comparing with pure mathematical graphs, a graph which shows the connectivity relationship of actual structural elements is very simple, because it is obtained in order to analyze a structure easily. For example, the number of lines which are connected to a node is restricted within a definite number for any graph, though it is unrestricted for a general graph. A graph which represents a plate structure divided into finite elements has a lot of nodes but the number of lines to a node is, in general, less than ten in order to reduce the occurence of numerical errors. For any framed structure the number of lines connected to a node cann't be increased infinitely, because it is restricted by the purposes of the structure and also by the economy of the construction of it.

Today we can find huge civil engineering structures with more than ten thousands of nodes. In a glance they seem to be very complicated but the restriction is kept at any part of them as well as a very simple structure has. In the sense it can be said that a graph corresponding to a civil engineering structure has very simple topology.

In the past studies of network analogy of sturctural analysis, only a part of graph theory is introduced and applied. Framed structure has just the same configuration as a network and the first application of network theory to the field of structural analysis was done to it.[1] It consists of members and joints, and end-nodes of a number of members are connected to a joint.

These relations between structural elements are invariant against any physical properties of the system and they are the topology of the whole system, itself. Furthermore, the desires of the matrix-form representation of the relaionships led to following matrices which represent the topology of any graph.

Branch Node Incidence Matrix

Node-to-datum Path Matrix

Branch Mesh Matrix

Basic Cut Set Matrix

They express only the connectivity relations of elements in a graph. That is, in order to understand a graph they are not sufficient and some other definitions and concepts of graph theory should be introduced in order to explain and clarify the following chapters. These definitions are used as the usefull tools for the new strategy of reducing the bandwidth of stiffness matrices of structural systems with various types of configurat-ions.

A linear graph, G, is a configuration which consists of only nodes and lines. A graph with "n" nodes and "m" lines is expressed by G(n, m).

In the graph theory a lot of general properties of any types of graph are investigated and we use only a part of them. Followingly, the author explains some concepts in the graph theory.[47-53]

(1) Distance (denoted by "d")

The distance between two nodes, A and B, is defined as the number of lines which locate on the shortest path connecting them. Thus, the distance between nodes, 3 and 5, in Fig. 3-1 is obtained as

$$d(3, 5) = 2$$

(2) Degree (denoted by "deg.")

Degree of a node is the number of lines which are connected to the node. For the graph in Fig. 3-1, the degree of node-3 is equal to 3. As every line has two terminal points and they are connected to different two nodes, we have following equation for a graph, G(n, m).

$$\sum_{i=1}^{n} (\text{the degree of the i-th node}) = 2m \tag{3-1}$$

(3) Diameter and Radius (denoted by "d_0" and "r_0", respectively)

For a graph, G(n, m), the longest distance among the shortest pathes from the i-th

node to all the other nodes is denoted by d_i.
This operation is repeated for all nodes included in the graph, and we obtain the result,

$$(d_1, d_2, \ldots, d_n)$$

Among them, the largest and the smallest values are called its diameter and radius, respectively. An example is shown in Fig. 3 1. In general, following relation exists between the number of nodes, diameter, and radius.

$$n \geqslant 2r_0 \geqslant d_0 \geqslant r_0 \qquad (3\ 2)$$

(4) Complete Graph (denoted by G_c)
If the distance between every two nodes is equal to 1, the graph is called to be a complete graph. In other words, every two nodes in the graph are connected by a line and the diameter and the radius are equal to 1. Between the number of nodes and lines of a complete graph, we have a relation;

$$m = n(n\ 1)/2 \qquad (3\ 3)$$

A triangle with 3 nodes and 3 lines is one of simple complete graphs and it is denoted by $G_c(3, 3)$. An example with 5 nodes is given in Fig. 3 2.

(5) Complement Graph (denoted by $\bar{G}(n, m)$)
The complement graph, $\bar{G}$, of a graph, $G(n, m)$, has the same number of nodes in G and has lines which connect every two nodes that are not connected in G. $\bar{G}$ is obtained by use of G and its complete graph, G_c.

$$\bar{G}(n, m) = G_c(n, n(n\ 1)/2) \quad G(n, m) \qquad (3\ 4)$$

The relation between G_c and G is presented in Fig. 3 2.

3–3. Graphical Representation of Structure

Actual structural system at the stage of analysis by using the matrix representation is considered to be transformed into equivalent network.

If the system is a truss structure or a framed one, the original system expresses the network, itself. Thus, the graph which presents the same topology of the original one is easily obtained by removing all the physical properties. That is, joints and members of a truss or a frame are represented by nodes and lines, respectively. Even if the structural element of the system has three dimensional properties, it is expressed at most by one-dimensional element, i.e. a line. That is, the graph presents only the original connectivity relationship, i.e. the topology of original system.

The most usual and useful treatment of continuous media for matrix analysis is the subdivisions of the object into a number of finite elements. That is, in order to remove the difficulties of treatment of the continuous media as a whole, it is divided into a number of small-sized elements whose physical characteristics can be approximately measured and, also, whose behaviour can be sufficiently expressed at some representative locations, and the characteristics of the system, itself, are represented as a connection of the properties of the finite elements. In the method, all of the physical properties are transmitted through the representative locations on the media. A finite element is obtained by connecting these representative nodes and usually the configuration is triangular or quadrilateral. If the element is a triangular one, it has three nodes and three lines which construct the boundary of the element. As shown in Fig. 3 3, every node of it has physical connectivity with other two nodes and the relations are represented by the three lines on the boundary. This indicates that the topology of the triangular element can be expressed by the graph, $G_c(3, 3)$, which has three nodes and three lines. (See Fig. 3 3)

In the case of a quadrilateral element, it has only four nodes at four edges and four lines which make up the boundary of the element. But the physical relaiton between nodes suggests that a node is influenced by the other three nodes. This is observed from the calculated stiffness matrix of the element being presented in Fig. 3--3. From this fact, the topology of the element can be expressed by a complete graph with four nodes and six lines, $G_c(4, 6)$. (See Fig. 3 3)

This completeness is restricted only in any element and a graph which presents the whole system is, of course, not a complete graph but has less lines than the complete graph should have.

If the actual topology of continuous media is secured in the analysis, we have to define some physical properties of lines, surfaces and volumes. But, as far as we use a digital computer as a tool for calculation, we must define the behaviour of the system at a number of representative nodes. In the sense, the topology of a continuous media may be sufficiently held in the graphical model in the previous sentences.

Summarizing them, any structural system can be drawn as a graph which shows the topology of the model for the analysis.

3–4. Topological Considerations on Bandwidth Problem

As described in the introduction, the minimum bandwidth at the state of optimum nodal ordering is decided only by the topology of the system.

If a structural system is drawn by a graph, G(n, m), whose nodes are ordered arbitrarily, we can make a branch node incidence matrix which is denoted by A (m × n) matrix and which gives whether an arbitrary pair of nodes are directly connected.

Dividing the matrix into as many column matrices as the number of nodes, it is shown as

$$A = \left[A_1, A_2, \ldots, A_n \right] \qquad (3\text{-}5)$$

Then, the configuration of the original graph may be presented by following equation instead of drawing the system.

$$K = \begin{bmatrix} A_1^tA_1 & A_1^tA_2 & \cdots & A_1^tA_i & \cdots & A_1^tA_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_i^tA_1 & A_i^tA_2 & \cdots & A_i^tA_i & \cdots & A_i^tA_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_n^tA_1 & A_n^tA_2 & \cdots & A_n^tA_i & \cdots & A_n^tA_n \end{bmatrix} = A^tA \qquad (3\text{-}6)$$

In the equation, we obtain plus integer numbers at main diagonal elements and minus or zero at off-diagonals. The former indicates the number of lines which are connected to the node, i.e. the degree of the node. The latter presents that the two nodes are directly or not connected each other, respectively. They are summarized as follows.

$$\begin{cases} A_i^tA_j = 0, & \text{if } d(i,\ j) > 1 \\ A_i^tA_j \neq 0, & \text{if } d(i,\ j) \leqslant 1 \end{cases} \qquad (3\text{-}7)$$

, where d(i, j) means the distance between the i-th and the j-th nodes. And we have a relation for every row or column.

$$\sum_{i=1}^{n} A_i^tA_j = \sum_{j=1}^{n} A_i^tA_j = 0 \qquad (3\text{-}8)$$

By the appropriate reordering of nodes, the maximum value of d(i, j) is decreased, and when the value is minimized, the nodes are ordered in optimum state.

d(i, j) = 1 indicates that the both nodes are directly connected each other. If a graph is given, the number of lines included in it is a definite number and the existence of a line is expressed by d(i, j) = 1. (See Fig. 3-4). Thus, the ratio of the number of nonzero elements to the total elements in the upper triangular matrix of K is constant in any state of nodal ordering and the ratio, ρ, is given by

$$\rho = 2\,\frac{(n+m)}{n(n+1)} \qquad (3\text{-}9)$$

in which n and m are the number of nodes and lines, respectively.

If the half bandwidth of K is decreased to p by any proper ordering of nodes, the ratio, ρ', is given by

$$\rho' = 2\,\frac{(n+m)}{p(2n-p+1)} \qquad (3\text{-}10)$$

The half bandwidth is defined as to include the main diagonal element.

If a graph is a complete graph, $G_c(n)$, all elements of its K are, of course, non-zeros and the half bandwidth is equal to n. In this case, the ratio, ρ, is equal to 1.

The diameter, d_0, and the radius, r_0, are also equal to 1 for the graph. And the ordering of nodes is, of course, arbitrary for a complete graph.

If only one line in a complete graph is removed, d_0 is increased by one, but the radius, r_0, does not change its value. Therefore,

$$d_0 = 2, \; r_0 = 1$$

In this case, the degrees of both nodes which were the terminal nodes of the removed line decrease by one and those of the other nodes don't alter. Thus, in the new reordering of nodes in the graph we know that only one element is appeared to be zero in the upper trianglar matrix of K and we should place it at (1, n)th element. By this treatment, the half bandwidth decreases by one.

In general, graphs showing actual structural systems are incomplete graphs and K of them include a lot of zero elements.

Even from above examples, we can suppose that the concepts of diameter and degree may play a part of the most important roles of the research for the optimum ordering of nodes.

Before considering complex graphs, the author presents very simple two examples which are called tree graphs. Using these typical and simple examples the factors which influence the bandwidth may be clarified.

Fig. 3–5 is an example of a tree with one centre of lines. The centre means a node whose degree is more than two and to the node more than two lines are connected. The characteristic of the graph is that the centre is complete, but the others are with deg. = 1. For the graph we know that $d_0 = 2$ and $r_0 = 1$. The optimum numerical ordering is easily obtained by experiences and is shown in the same figure. In this case, the number of the centre is decided by the equation as follows.

$$\text{Number of centre} = [n/2] \text{ or } n - [n/2] \qquad (3\text{–}11)$$

In the equation [ ] indicates the Gaussian symbol. The labeling of the other nodes are arbitrary, and K of this graph is also presented as K_I in Fig. 3–5. For this example, H. B. W. = 5, where H. B. W. is the abbreviation of the half bandwidth.

The second example is a series of lines and is presented in Fig. 3–5. If the graph contains n nodes, it can be expressed by G(n, n−1) and it includes the least number of lines among n-node connected graphs. K_{II} in Fig. 3–5 shows K of the graph and it is obvious from this example that the fact, H. B. W. = 2, indicates the graph being the simplest one with minimum bandwidth. This is because a graph with H. B. W. = 1 is no more a connected graph but a group of nodes without lines. Furthermore, even if a column or row has only one non-zero element which locates, of course, on the main diagonal, the graph is no more a connected one and the graph consists of two independent subgraphs. This type of graph is not treated here, because they can be treated just as two graphs expressing independent two structures from the stand point of structural analysis. In the above example, the diameter is equal to (n 1) and the graph is the diameter, itself. The degree of both ends is equal to one, and all the other nodes have the degree of 2.

These two examples appreciate the the correctness of the considerations for the bandwidth which are done for a complete graph and a graph which is obtained from G_c by removing only one line.

More experiences has a man for the ordering of nodes, better results can be obtained without using any tool like the digital computer. On the other hand, the algorithms of the bandwidth reduction founded up-to-date are, of course, useful and require less computation in general, but for some kind of structures we cann't expect sufficient results. Sometimes, the experiences of the analysts may lead to better results than the use of the algorithms can do. This is caused by the lack of the topological considerations in the proposed algorithms, as described in the introduction of this chapter.

Above considerations to find out some factors which influence the bandwidth of a system are achieved for a special system, i.e. tree system.

But general civil engineering structures contain a lot of meshes. Truss structures, framed structures and also plate-like structures may be expressed by use of a kind of meshes .

Treating some simple mesh graphs to give its optimum numerical ordering clarifies that the factors obtained by treating tree graphs, i.e. diameter and degree, are very important for them also, but there exist some differences.

For tree graphs the shortening or elongation of a line is free and it does not influence the other structural elements. But by the change of the length of a line in a mesh graph, the whole configuration of it is necessarily twisted or bent, even if the original configuration is a planner one. This fact is occured from the biggest difference between mesh and tree. That is, in a mesh a number of different pathes exist between two nodes but in a tree graph there exists only one path.

Followingly, some mesh graphs are investigated in order to find other important factors which are useful for the bandwidth reduction.

If a system is a rectangular shaped plate and it is devided into finite elements with same lengthes as shown in Fig. 3 6, we can easily order the nodes in optimum state from our experiences. But if its boundry has an irregular shape or it is divided into finite elements whose lengthes and shapes are all different each other, the ordering of nodes becomes very complicated.

The latter case will often occure when we treat plates with cracks or holes and it is expected that the stress concentration will appear near the irregular places.

The difficulties of nodal ordering for these cases seem to be occured from the irregularity of the boundary configuration and also from the non-uniform distribution of nodes. The non-uniformity of nodal distribution leads to the inequality of the length of boundaries of finite elements. For the bandwidth problem, the actual length is useless but only the connectivity between nodes is needed for our purpose.

In order to distinguish the difference between the original graph and the transformed graph with graphical distance they must be actually drawn. Between them there exists no difference of their topology but only the differences of their actual configurations. As far as we treat graph only, the modification of the configuration is allowed and has no influence to the topology. That is, as far as the new graph keeps the original topology, we may elongate or shorten the length of line.[54-58] By this operation, we can stretch, twist and reduce a part of the graph or, of course, whole of the graph, and replace it into a configuration by which we can guess the bandwidth.

Showing some examples, the author explains the operation described in above sentences and discusses the merits for the optimum numbering of nodes.

The first example is a simple graph with only one mesh. This example is presented in R. J. Collins' paper, and the mesh contains only 16 nodes.[37] The diameter is a half length of the mesh.

$$d_0 = 8$$

The optimum ordering of nodes is easily obtained and

$$\text{H. B. W.} = 3$$

Similar example is given by R. Rosen.[32]

The second example is a plate-like structure. A plate is divided into finite element with triangular configuration as shown in Fig. 3-7. As this plate has a hole at the centre, the elements near the hole are smaller than those near the outer boundary. Observing the figure we notice that the number of nodes on the inner boundary is equal to that of the outer boundary. Equating the length of lines on both boundaries yields to equating the total lengthes of both boundaries. A quarter of the graph for the plate can be rewritten into a new graph as shown in Fig. 3-7-b, and the whole of the original graph may be expressed by Fig. 3-7-c. That is, the plate with inner and outer boundaries can be transformed into a kind of a pipe which has two boundaries at both ends. In this example, lateral lines in Fig. 3-7-b have the same length and inclined lines have different length. Using this operation, original system is replaced by systematic one whose nodal distribution is uniform and whose boundary configuration is very simple. In the new configuration the location of the diameter is easily obtained and along the direction of the diameter the graph is stretched and we can suppose that the half bandwidth can be decreased up to 11, though the value cann't be found in a glance of the original configuration.

Fig. 3-8 is a L-formed plate and it has 1566 triangular elements and 747 nodes. This example has many elements but it has very simple topology and we can easily transformed it into Fig. 3-9. This example has only one boundary and it is kept in the new configuration.

Our experiences teach us that the half bandwidth of the system is equal to 26 from this new configuration. And the value may be the minimum one.

We will give some consideraitons for this new graph given in Fig. 3-9. Every line combines two nodes which are included in neighbouring two nodal columns. When we give nodal numbering for this new graph, we give the ordering in the nodal column from the right to the left and also from the top to the bottom in a column. Then, we can observe that the maximum difference of nodal numbers which are connected by a line occures at the nodal column which includes the largest number of nodes. This suggests that the maximum width of a graph gives the maximum bandwidth. Thus, if we treat a graph in accordance with above graphical representation, we should reduce the number of nodes in a nodal column in the new configuration as possible as it allows. At the same time the inclination of a connecting line is also influent on that value of the bandwidth.

These operations are used for the cases of irregular configuration of boundary and it is sufficiently related and discussed in the following chapters.

Fourth example is a very simple framed structure with 19 joints and 34 members, and the topology is just kept in the graph in Fig. 3-10-a. If we introduce d = 1 for every line and equate their lengthes, the graph is bent and is shown in Fig. 3-10-b. This new configuration is a kind of ring form and it is similar to the deformed one of the first and the second examples. And we know that H. B. W. = 5. This example is examined by E. Cuthill and J. McKee, and R. Rosen,[39] and their results are just same as the value which is obtained here. Investigating the configuration of this example, the node with minimum degree should be labeled by the initial number and the whole configuration shows convex. These factors induce the best result for their algorithms.

Fig. 3-11 shows a graph of plate-like structure and its topological equivalent configuration which presents the location of diameter evidently. Cuthill and McKee compaired their result with Rosen's result.[39]

From these four examples we can obtain following items.

1). The configuration of a system for the ordering of nodes needs not to be drawn in the same dimensional space but may be transformed in the space with higher dimensions than the original one. Example 2 and 4 show the increasing of dimensions for spaces where new configurations are drawn, and Example 1 and 3 present that the new ones are drawn in the same dimensional spaces as the originals are figured.

2). We can have the possibility of showing the bandwidth in optimum ordering of nodes by the maximum width of a graph which cann't be reduced any more. For the purpose, we have to introduce a new coordinate system whose one axis presents the magnitude of bandwidth. In the new coordinate system, some restrictions are needed for the rearrangements of lines and nodes.

3). For the drawing of a graph in the coordinate system, the existence of symmetric axis of original structure will be a useful tool which can be used for imaging the vague outline of transformed graph.

4). It can be said that the number of boundaries of the original structure is also very important factor for the appreciation of the transformed configuration. Transformed structure has to keep the same number of boundaries as the original has. In the sense, a plate with a hole and a pipe with both ends opened are classified in the same category. They have the same topology. Thus, we can suppose that a plate with a hole may be transformed into some typical configurations as shown in Fig. 3 12.

These treatments of original graphs are done only for the systems with non-uniform distribution of nodes. After the above operation in order to make the nodal distribution uniform, the configuration may have irregular boundaries. For systems with uniform nodal distribution the above procedure is, of course, useless.

In accordance with the above considerations, the width of a graph should be reduced as possible as it allows. That is, the branches of the graph should be filed along the longitudinal axis of the new coordinate system, if another axis is used to show the magnitude of bandwidth. This treatment coincides with the technique of Sequential File Method which the author proposed already for tree systems and is shown in Chapter 5 in this thesis.

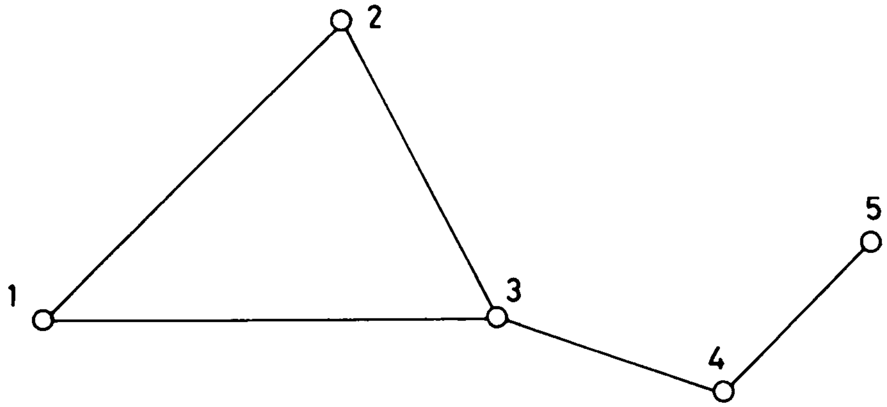
In the following chapter, the author introduces a new coordinate system which can express the bandwidth of any graph by one axis. The coodinate system satisfies some restrictions which are concerned to item 2.

3–5. Conclusions

The investigations in this chapter were mainly proceeded to make clear the influences of topology of structures to their bandwidthes. We can conclude that the topology of the model of a structure is its bandwidth, itself. That is, the minimum bandwidth of a structurs is one of its original characteristics. Thus, in order to find out it we have to, of course, investigate the topology and use it.

If the topology governs the bandwidth, the chracteristic of topology should be usefully taken into consideration to find out the basic strategy for bandwidth reduction method.

As far as we treat original configuration, the outward appearance of it will not allow to find out its minimum value of bandwidth. It concludes that the original configuration should be modifed without changing the original topology into a new configuration which shows the characteristic of the system clearly. That is, a system should be transformed into a new one from which we can easily find out the minimum bandwidth. Therefore, it is necessary that a space on which a structural system is drawn must be defined, and also that how to draw a system on a space must be found out.



Maximum Distance from Each Node in Above Graph

Initial Node	Max. Distance
1	3
2	3
3	2
4	2
5	3

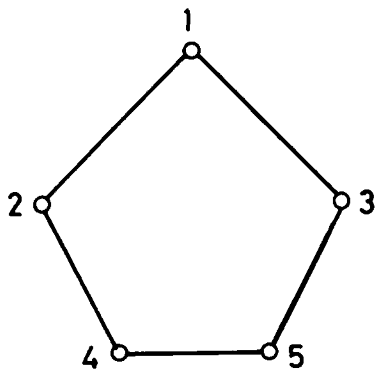
Diameter (d_0) = 3

Radius (r_0) = 2

Center of Graph : Node 3

Maximum Degree = 3 (at Node 3)

Fig. 3 1 Diameter, Radius of Graph and Degree of Node

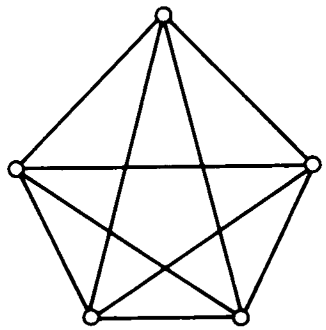


Graph : G(5, 5)

$d_0 = 2$

$r_0 = 2$

H. B. W. = 3

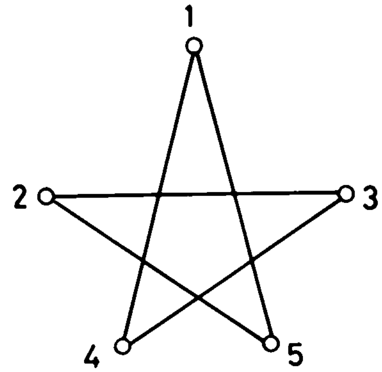


Complete Graph : G_c

$d_0 = 1$

$r_0 = 1$

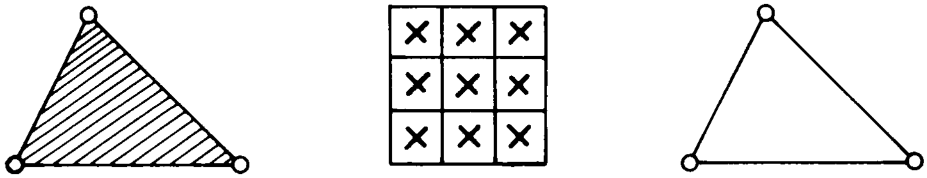
H. B. W. = 5



Complement Graph : $\overline{G}$

$$G_c = G + \overline{G}$$

Fig. 3–2 Graph, Complete Graph and Complement Graph

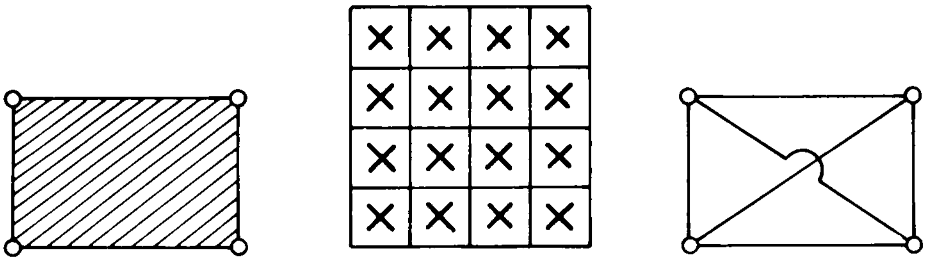


(a) Triangular Element

(b) Stiffness Matrix

(c) Graph for Triangular Element

(I) Triangular Element and Its Graphical Representation



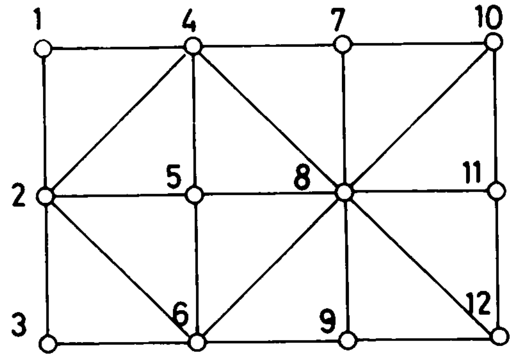
(a) Rectangular Element

(b) Stiffness Matrix

(c) Graph for Rectangular Element

(II) Rectangular Element and Its Graphical Representation

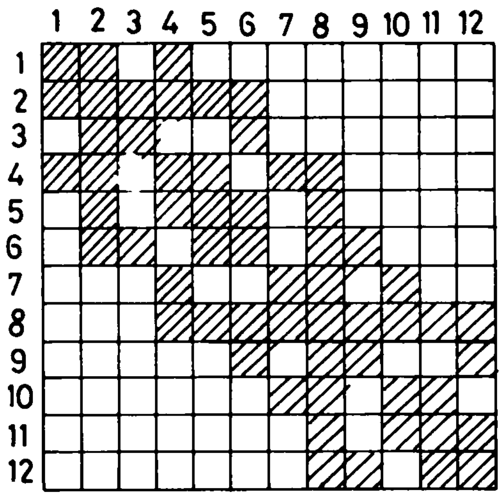
Fig. 3-3 Structural Elements and Their Graphical Representaions



A : Branch Node Incidence Matrix

$$a_{ij} = \begin{cases} 1 : & \text{the i-th branch is positively incident on the j-th node} \\ 1 : & \text{the i-th branch is negatively incident on the j-th node} \\ 0 : & \text{the i-th branch is not incident on the j-th node} \end{cases}$$

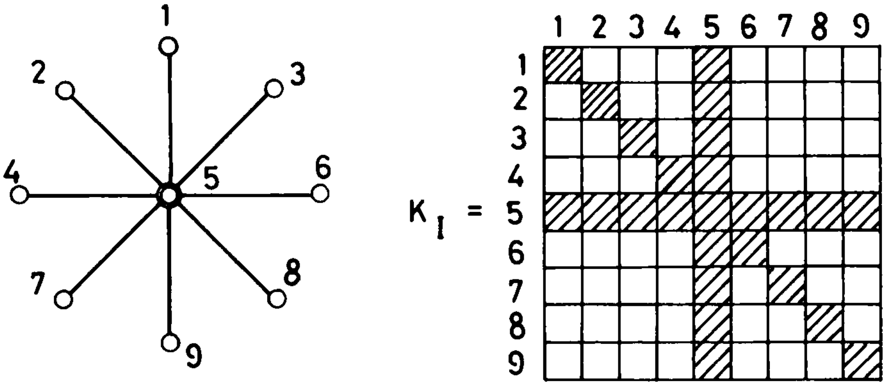
$K = A^t A =$



$$A_i^t A_j = 0, \quad \text{if} \quad d(i, j) > 1$$

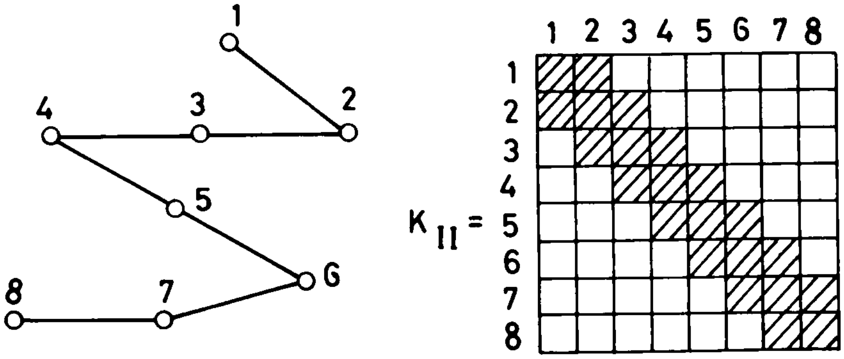
$$A_i^t A_j \neq 0, \quad \text{if} \quad d(i, j) \leqslant 1$$

Fig. 3–4 Branch Node Incidence Matrix and Relation between Nodes



(a) System–I

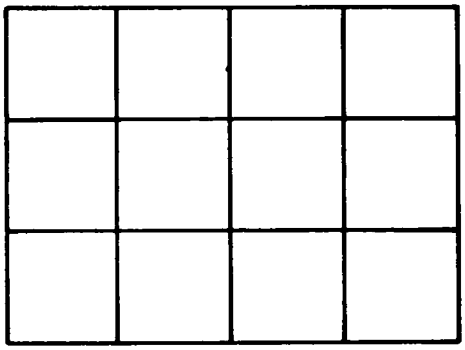
H. B. W. = 5



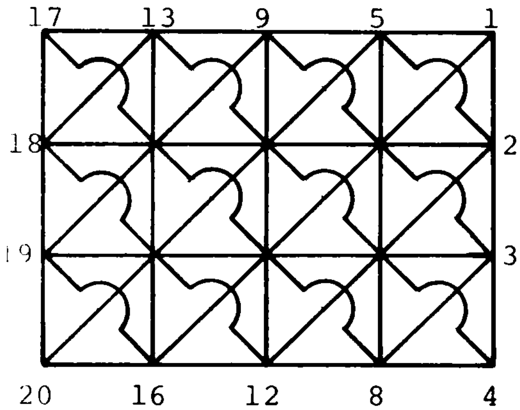
(b) System–II

H. B. W. = 2

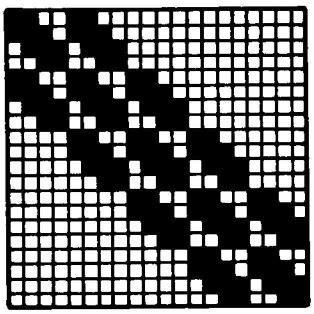
Fig. 3–5 Typical Two Tree Systems with Optimum Node-Numbering



(a) A Plate Structure with 12 Elements

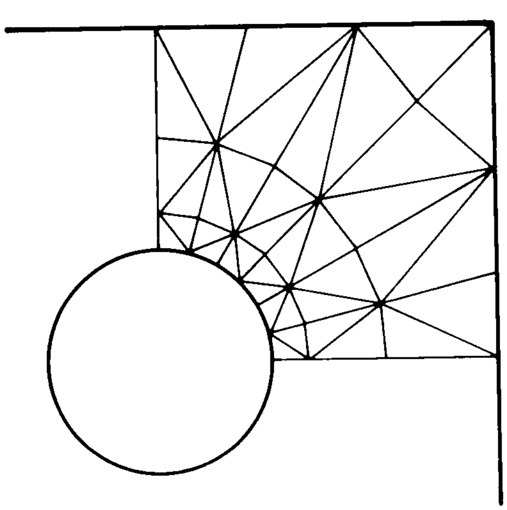


(b) Equivalent Graphical Representation of The Plate and Nodal-Labeling

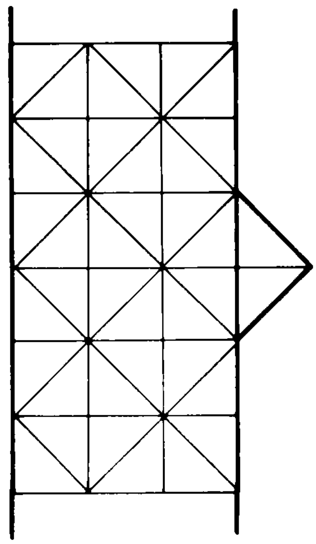


(c) Stiffness Matrix with Minimum Bandwidth (H. B. W. = 6)

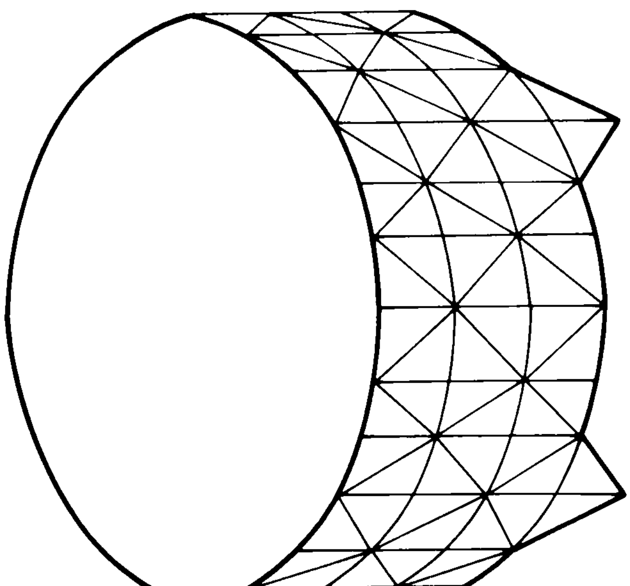
Fig. 3–6 Simple Example of Nodal-Labeling



(a) Original Plate Structure
Divided Into Elements



(b) Transformed Configuration
of A Quarter of The Plate



(c) Topologically Equivalent Configuration of Original Plate

Fig. 3–7 Topologically Equivalent Modification of Configuration of A Plate with A Hole

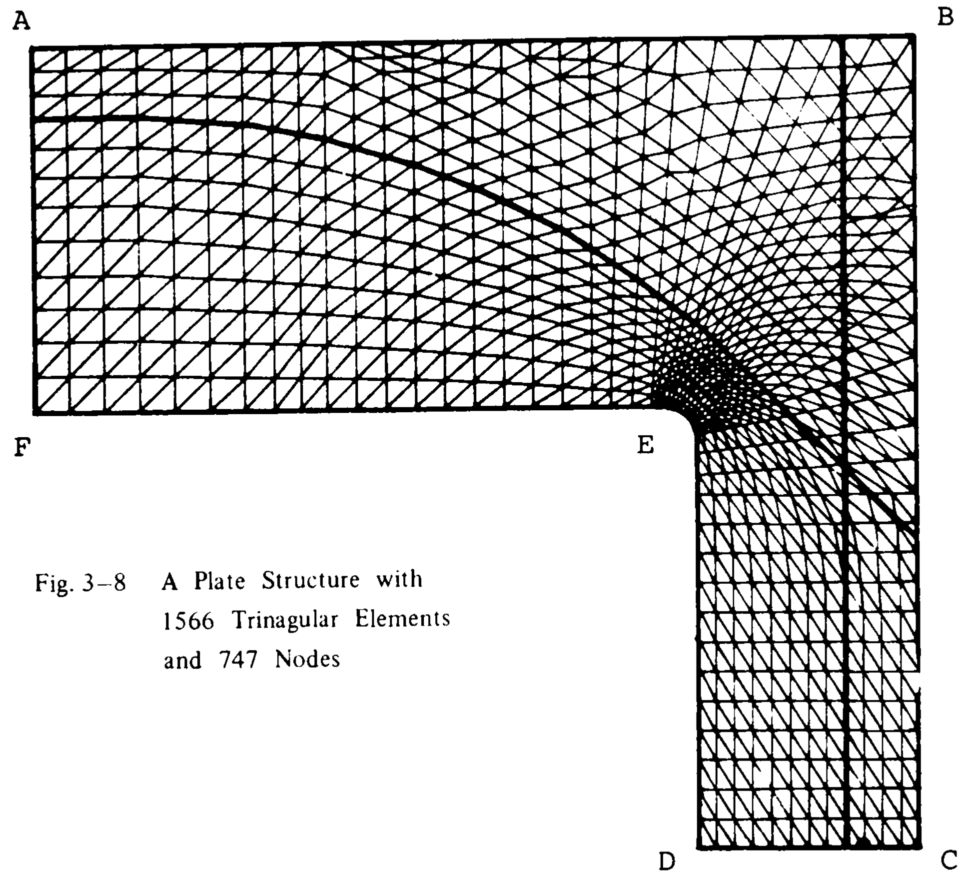


Fig. 3–8 A Plate Structure with 1566 Trinagular Elements and 747 Nodes

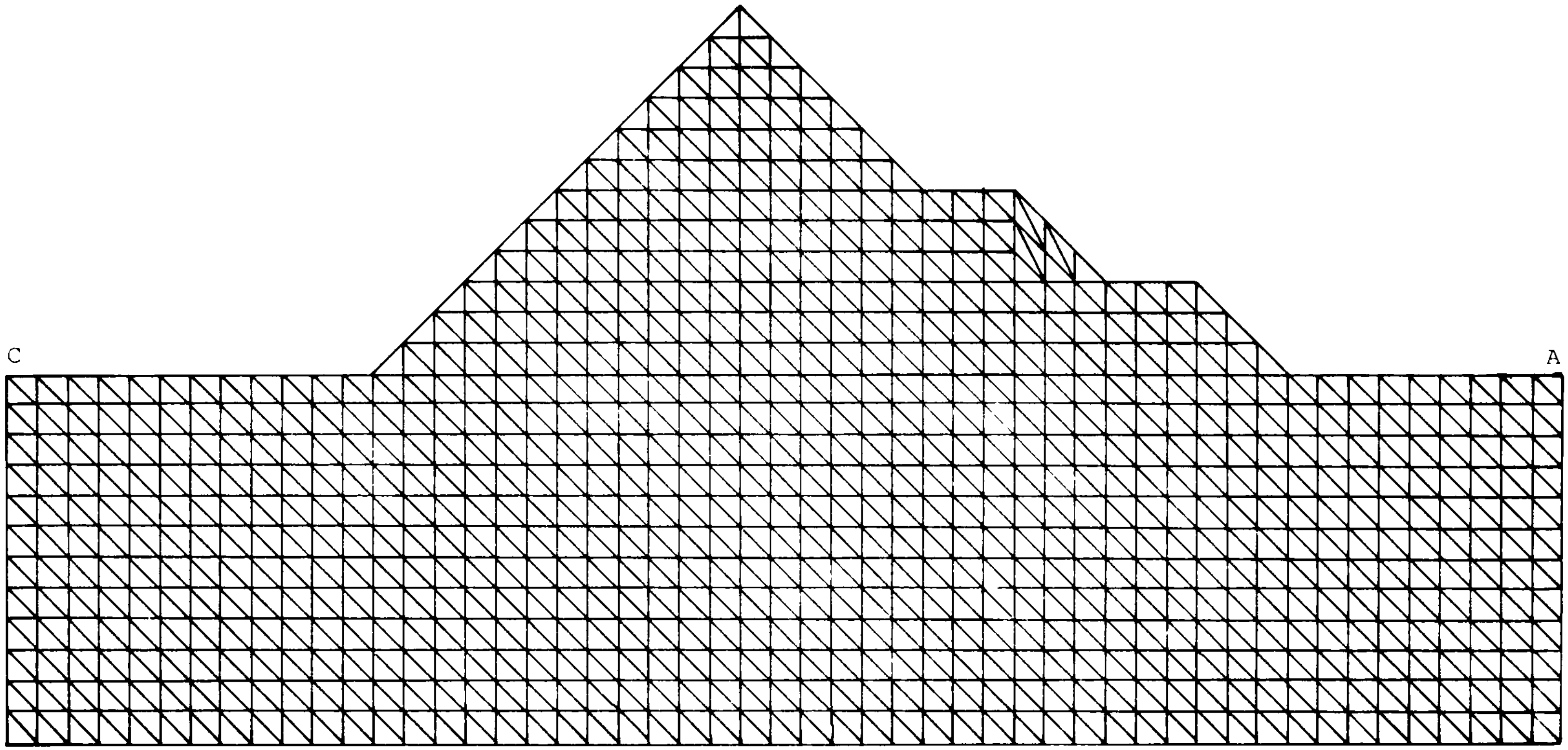
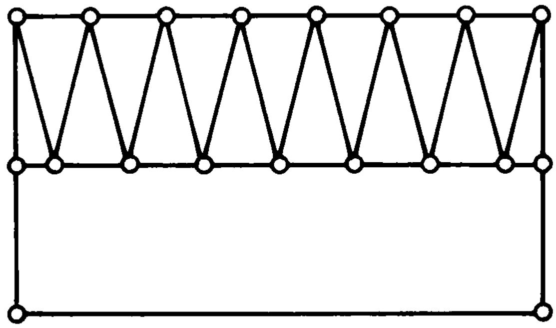
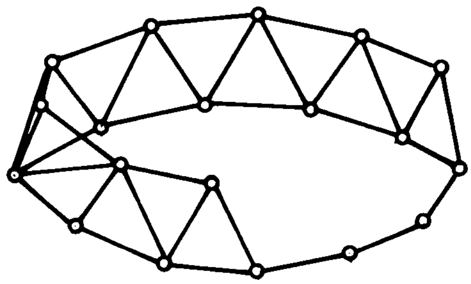


Fig. 3–9 Transformed Configuration of A Plate Structure (H. B. W. = 26)

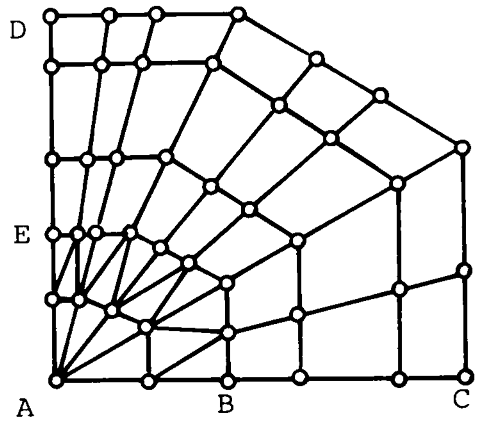


(a) Original Graph with 19 Nodes and 34 Lines

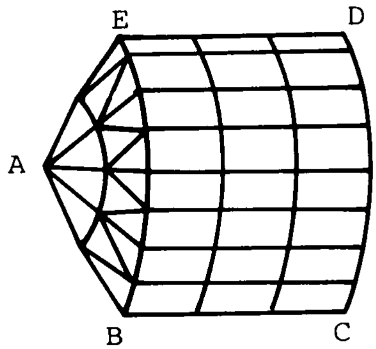


(b) Transformed Graph Composed of Lines with Same Length (H. B. W. = 5)

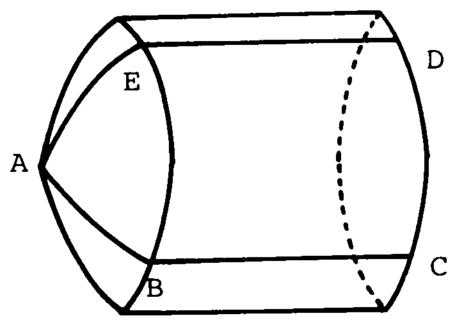
Fig. 3-10 Topologically Equivalent Modification of A Simple Graph



(a) Original Plate Structure

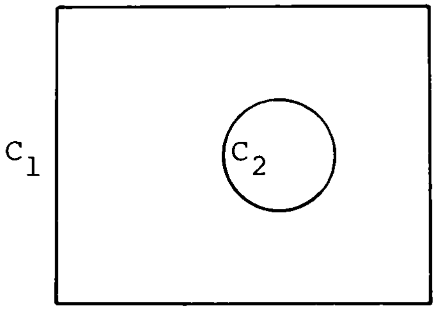


(b) Topologically Equivalent Configuration of The Plate

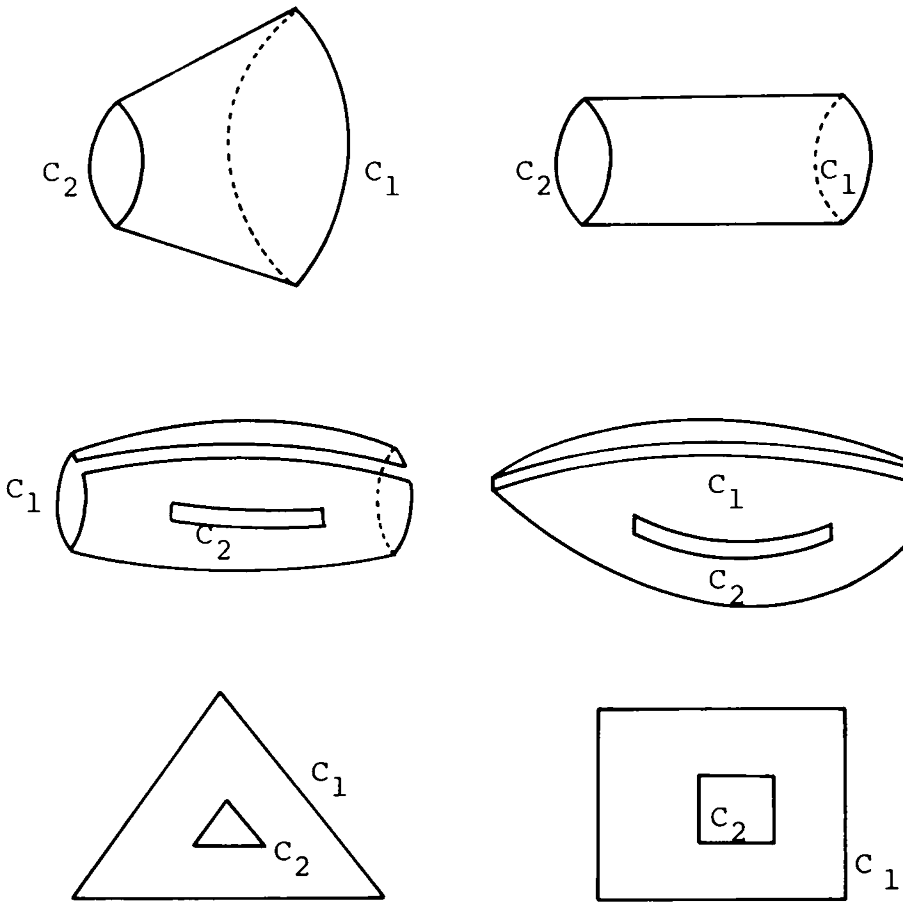


(c) A Part of A Shell Structure

Fig. 3-11 Topologically Equivalent Configuration of Plate Structure



(a) Original Plate Structure with A Hole



(b) Topologically Equivalent Configurations

Fig. 3-12 A Plate with A Hole and Possible Configurations with Same Topology

CHAPTER 4
DEFINITION OF FILING FIELD

4–1. Introduction

In the conclusions of Chapter 3 the necessity of a space in which a graph is drawn is led. Thus, this chapter contributes to define the space which is called "a filing field".

Filing field is a kind of coordinate system in which a given graph is arbitrarily drawn with keeping its original topology and whose one axis should be used to measure the bandwidth of a drawn graph.

The drawing of a graph in the field corresponds to homomorphic mapping. The mapping keeps the connectivity relationship of the original graph and the image holds the original topology. But the cofiguration of the image may be different from the original one as shown in Fig. 3–12. It shows some images of a plate structure with a hole. Six configurations contain the same topology as the original plate has. From their outward configurations they cann't be classified into a same category of configurations. But by some proper operations of stretching, twisting and elongation one of them can be fitted to other configurations. For example, a cylindrical shell may be back-transformed into an original plate by the elongation of one end.

These operations are to find out simple configuration of a given system which clearly shows the minimum bandwidth. Thus, the number of dimensions of a graph is not important for the bandwidth problem, and as shown in Fig. 3–12 an original configuration may be drawn in a space with the same or the larger number of dimensions. That is, for some cases the correspondence between original graph and its image can be appreciated by the number of boundaries. In Fig. 3–12, four configurations are three dimensional ones and the others are drawn in two dimensional field.

In this chapter, two-dimensional and three-dimensional filing fields are introduced and defined. And the author investigates the characteristics of the respective fields and he mentions the relations between the both fields.

4–2. Two-Dimensional Filing Field

As far as tree graphs are treated, the interchanges of a number of nodal sequences which are gathered to a node may be freely allowed in order to find out a new transformed tree graph which clarifies its minimum bandwidth. Or, some simple mesh graphs are also similarily treated as tree graphs. These graphs can be drawn on a plane.

For the mapping of these simple structures the author defines "two-dimensional filing field".

As mentioned in the introduction, one axis of the field must coincide with the bandwidth of a mapped graph. That is, by measuring the value of the axis it should be judged whether the image can give good result (i.e. the minimum bandwidth). Thus, the field must admit any mapping of a configuration. That is, the field must be able to show any image of a given configuration with bandwidth between its maximum and the minimum values.

To satisfy the coincidence of the bandwidth of a image and the value of one axis, following restrictions are given to the two-dimensional filing field.

1). A two-dimensional surface is divided by vertical and lateral lines which locate with same distance and the crossing points of the lines are prepared for the locations of nodes of original graph. At the crossing point, only one node can be placed at the mapping of the original graph. After the mapping, the vertical and the lateral nodal arrays are called "nodal column" and "nodal row", respectively.

2). Allowable directions of a connecting line are restricted to be lateral, vertical, and obliquely descending to the right or ascending to the left. A line within these angular restrictions is called to be "positive".

3). A line can connect only two nodes which belong to a same nodal column or, also, to two neighbouring nodal columns.

The field which satisfies these three restrictions is called the two-dimensional filing field and it is illustrated in Fig. 4-1.

In the stage of giving numerical ordering to the nodes in the mapped graphs, we give it from the top node to the bottom in a same nodal column and, also, from the right-side nodal column to the left, successively. By this nodal labeling, the maximum difference of two nodal numbers appears at a lateral line which connects two nodes in neighbouring two nodal columns with maximum number of nodes in the field. If the number of nodes in a column is equal to n, the half bandwidth is expressed by following equation.

$$\text{H. B. W.} = n + 1 \qquad (4\text{-}1)$$

, where H. B. W. is the abbreviation of "half bandwidth".

Among above restrictions, the first item makes the distribution of nodes uniform for every part of the graph, and the second condition equates the bandwidth to the difference of nodal numbers. The third restriction suggests that every two nodes which are directly connected each other must be placed in a column or in neighbouring two nodal columns.

Here, we consider to remove the second restriction. By removing the condition, the maximum half bandwidth will appear at a line which connect two nodes in neighbouring columns and has the steepest negative angle descending to the left. If the left-side column has less or equal to the number of nodes than the right-side column has, the former can be arbitrarily translated upward within the length of the latter. If this operation sets all of the lines within allowable directions, the equation of H. B. W., eq. 4-1, is available for this case, too. If not, we decrease the negative angle as possible as it allows by the operation. If there leave some lines having negative angles, the left-side nodal column is translated upward till all of the lines will be set in allowable directions, that is, a line with steepest negative angle should be set in lateral direction by the vertical translation of the left-side nodal column. If the top of the left-side nodal column is higher than that of the right-side one by α, the above equation for H. B. W. is modified as following.

$$\text{H. B. W.} = N + \alpha + 1 \tag{4-2}$$

Thus, we can conclude that we must calculate the half bandwidth by using above equation, if we use the filing field without the second restriction for allowable directions of connecting lines. And, if we aim to decrease the H. B. W., we should pay attention to the maximum number of nodes in a column and also the maximum negative angle of a line

Filing field is only a tool for bandwidth reduction but not the method, itself. It explains any numerical ordering of nodes which we can arbitrarily give for a graph.

4–3. Three-Dimensional Filing Field

The necessity of a three dimensional filing field comes from following reasons.

1. The topology of original graph makes difficult and troublesome to draw it in two dimensional filing field.
2. When original graph is transformed into a graph with same distance for a number of lines in order to guess and image the outline of the transformed one, it is often occured that the configuration is necessarily drawn in three dimensional space.
3. In two dimensional field, the numerical ordering in neighbouring two nodal columns follows from the bottom node of the right-side nodal column to the top node of the left-side one. Between these two nodes there is no line but an imaginary line may be set. Thus, we may set these two nodes by $d = 1$.

This filing field is set in three dimensional space. The coodinate system is shown in Fig. 4-2. Along the x-axis, any graph is stretched as long as it can be. Thus, the x-axis corresponds to the direction of the diameter of the graph. Every cycle (i.e. circle) with radius, r, corresponds to the nodal column in two-dimensional filing field, and the radius, r, is decided by the number of nodes which are included in a nodal column.

Then, the relation between the number of nodes in a cycle and the radius is easily obtained and is shown in Table. 4–1. In a cycle, every neighbouring two nodes are located by $d = 1$. Thus, the length of a cycle or the radius is linearly increased in accordance with the number of nodes in the cycle.

Any line which connects neighbouring two nodes appears as an arc line on the surface of the configuration. And a line connecting two nodes in a same cycle appears as a chord line of a circle. This suggests that the bandwidth depends upon the radius of circle in the three-dimensional filing field. And if we aim to reduce the bandwidth, we should make the radius as small as possible.

In order to keep three restrictions which are defined in the two-dimensional filing field, we continue to investigate the properties of this new filing field.

Fig. 4 3 shows a very simple example of this filing field where a graph is already filed. The configuration is like a cylindrical shell. Every circle (i.e. cycle) contains the same number of nodes and its original graph in two-dimensional filing field may be drawn as shown in Fig. 4- 3. Upper and lower edges of original graph are displaced to hold the distance equal to 1. And the number of boundary is kept in the new graph. In this case, the graph is drawn on the exterior surface of the sylindrical surface. On the other hand, if we draw the graph on the interior surface of the cylinder, the graph is just presented in Fig. 4–3. Comparing these two cases, the nodal numberings are done in accordance with the arrows in the figures. And we notice that the directions of arrows are opposite for these cases. In the following of this thesis, the author mainly uses the former case. In these two cases, three restrictions in the previous section are kept, if we look at the new graph from the normal direction to the surface. And we can observe for this configuration that the lines of the boundary is straight along the x-axis. Thisis caused by following two reasons;

1). Every cycle contains the same number of nodes, and
2). The directions of all lines in the original graph shown in Fig. 4–3 are in the allowable directions which are described in the previous section.

This fact cann't be kept in general case. Every cycle may contain different number of nodes, and it leads to the occurence of different radius. At the same time, the line of original boundary cann't be drawn straight along the x-axis, but shows zig-zag line.

If we remove the second restriction at the stage of mapping on the three-dimensional filing field, the operation of drawing the outline of the transformed graph becomes easier than we draw the configuration with the restriction. But, there appear a number of lines which don't obey the condition of allowable directions.

For the two-dimensional field, we translate the left-side nodal column upward till the lines with negative angle are set to be lateral.

For the new filing field, we introduce the operation of twisting every neighbouring nodal cycles each other till the directions of lines are set in allowable directions. This operation is presented in Fig. 4–4. From these facts, we can know that the operation of translation in two-dimensional field is replaced by the twisting operation for three-dimensional filing field. The above operation of twisting the mapped configuraion induces the inclination of lines on boundary which are straight before the twisting. And as the smallest number should be labeled to the node on the boundary among the nodes in a cycle, the twist influences the bandwidth. If the left-side cycle is twisted against the right-side cycle by one nodal row, the bandwidth is increased by one. Thus, we can obtain following equation for the bandwidth of a graph.

$$\text{H. B. W.} = n + \beta + 1 \tag{4–3}$$

, in which n is the number of nodes in a cycle, and β means the left-side cycle being twisted by β nodal rows against the right.

From this equation we conclude that the newly drawn graph should have the least number of nodes in any cycle and, also, it should contain the least number of twisting between neighbouring two cycles, especially for the cycles with maximum number of node among them, in order to reduce the bandwidth.

4–4. Relation between Two-Dimensional and Three-Dimensional Filing Fields

Any graphs can be filed in both filing fields, and they have not any original difference between them. They are introduced only from the convienience of the mapping operation of graphs.

The biggest difference is the space where they are defined, and two-dimensional filing field is defined in a plane, while the three-dimensional one is a spacial surface.

The length of the x-axis of the three-dimensional filing field which is occupied by the mapped graph corresponds to the lateral length of the two-dimensional field.

The half bandwidth of a graph is given by its height for the two-dimensional one and by its circle (or cycle) for the three-dimensional one.

Thus, we may conclude that the three-dimensional one is a modification of the two-dimensional one, i.e. the upper and the lower edges of the latter are placed by $d = 1$. But this can be said only for certain cases in which the upper and the lower edges of two-dimensional field show exact boundary of the original graph.

In general, the both edges may not express the boundary. For example, if we treat a graphical model of a baloon, the original structure, itself, shows a boundary but it contains no boundary. And it can be drawn in two-dimensional filing field, and the both edges, of course, don't express the boundary. The drawn graph is just a configuration which is imaged by pressing the graph of a baloon on a plane and by extending one surface upward in order to remove the holding of two graphical surfaces. For this case, the three-dimensional field can show its merit and the original graph may keep its configuration in the new field after its mapping.

Thus, we can exactly conclude that the three-dimensional field is obtained from the two-dimensional one by connecting their upper and lower edges by $d = 1$ when they express the boundary of original graph or by restorating the original configuration of graph which is disappeared in the latter field.

The selection of the filing fields may be rendered to analysts who treat graphs, and they may select one which is convienient to clarify the topology and also which gives much informations for the reduction of the half bandwidth.

4–5. Conclusions

The bandwidth reduction method of any structural system is fundamentally divided into two themes.

The first theme is the definition of spaces which can admit any mapping of original system and whose one axis presents the bandwidth of the image. The second theme is how to map any structural system on the spaces.

In this chapter, the definition of the spaces is done and the author proposed two spaces which are quite different each other in a glance but are the same one.
They are called "two-dimensional filing field" and "three-dimensional filing field".

The selection of filing fields is rendered to analysts but it should be decided in accordance with the configuration of structural systems being treated.

Defining the filing fields, some restrictions are found. One of them is the restriction of allowable directions of connecting lines, but the restriction may be removed if much troubles are supposed to occure in treating very complicated system. In place of the restriction additional operation is needed after mapping of a system in the fields. For two-dimensional field vertical translation of nodal columns corresponds to it, and twisting operation must be done for the three-dimensional filing field.

After the definition of the spaces, there leaves how to map original structure on them and they are discussed in following chapters.

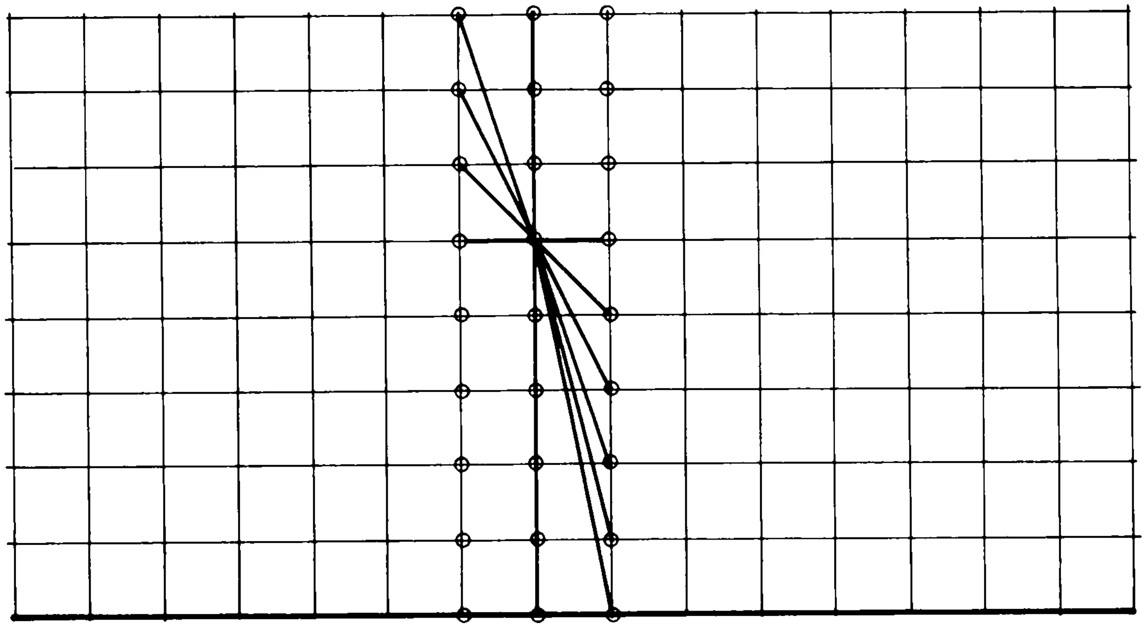


Fig. 4–1 Two-Dimensional Filing Field

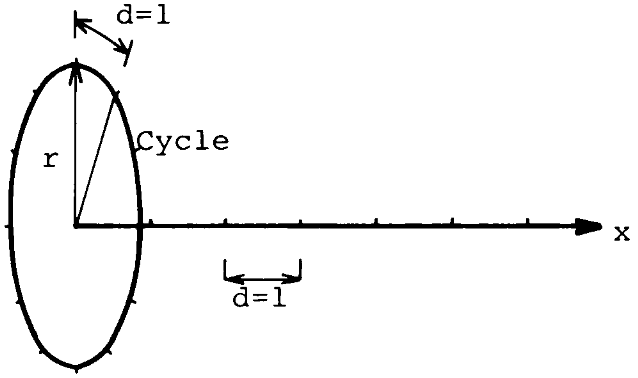
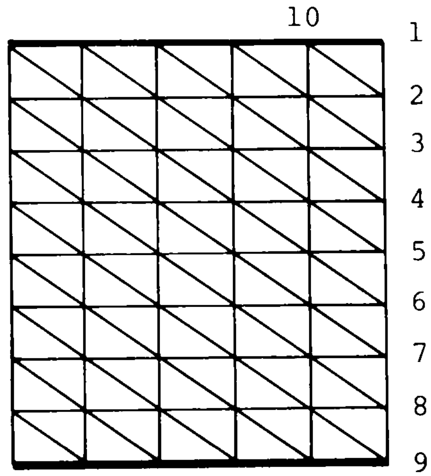


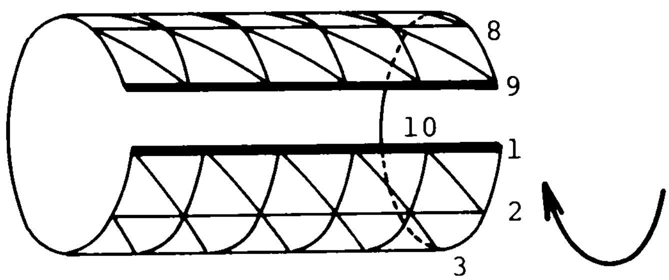
Fig. 4–2 Three-Dimensional Filing Field

Number of Nodes	1	2	3	4	5	6	.....	n
Radius(r)	0	1/2	3/4	1	5/4	3/2	.....	n/4

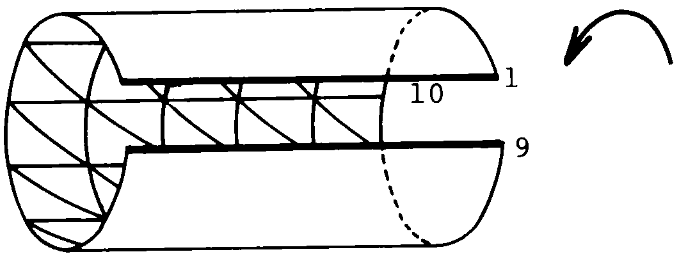
Table 4–1 Relation between Number of Nodes and Radius in Filing Field



(a) A Graph in Two-Dimensional Filing Field



(b) A Graph on Exterior Surface of Three-Dimensional Filing Field



(c) A Graph on Interior Surface of Three-Dimensional Filing Field

Fig. 4–3 Filing Fields and Ordering of Nodes

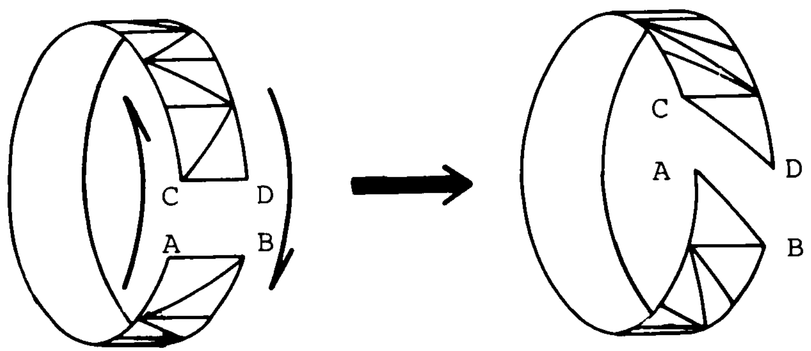


Fig. 4–4 Twisting Operation of A Configuration in Three-Dimensional Filing Field

CHAPTER 5
SEQUENTIAL FILE METHOD

5-1. Introduction

In this chapter the author treats a kind of statically determinate system (i.e. tree system) and he proposes a useful nodal labeling method which gives a minimum bandwidth.

Tree system was scarcely treated in past studies and we find only one paper by E. Cuthill.[40] She compared the results obtained by using Cuthill-McKee,[39] King[45] and Levy's algorithms. These algorithms cann't show their efficiency to tree system, because they can give good results for systems with convex boundary configurations. Tree system is thought to have just concave boundary configuration.

Sequential file method is proposed in order to reduce the bandwidth of tree systems which have distinguished property comparing with general structural systems.

We call systems being trees when they have no closed pathes. Thus, they are not statically indeterminate systems but a kind of determinate structures.

They show the configurations which can be imagined from the name, i.e. tree. Therefore, we find not so many kinds of actual structures corresponding to tree systems, but we often use tree systems as the analytical models in order to simplify the analysis.

Topology of a tree system is drawn by a tree graph which includes no meshes.

Tree graph corresponding to a structure is a kind of simplest graphs and it contains (n - 1) lines, if it has n nodes. Thus, the graph is denoted by G (n, n 1).

A tree graph includes the least number of lines which construct a connected graph and we know that the graph is divided into two subgraphs, if any line is removed from the original graph.

Another property of the graph is that only one path exists from any arbitrary node to another node, while there are many routes for mesh graphs. The maximum distance between every two nodes is decided by the number of lines which are included on the path connecting them. Thus, in the stage of giving numerical ordering to nodes, these two nodes may be labeled as to have, at most, the difference of nodal numbers equal to the maximum distance between them. Their relative location in the numerical ordering is decided by paying attention to this restriction, but for mesh graph relative location of every two nodes has to satisfy all of the restrictions which can be caused by the number of routes that can be found between them. These facts suggest that the numerical ordering for the latter is more difficult than that of the former.

Using the concepts of filing field as a tool for the bandwidth reduction, the graph should be laid on the field as long as it can be stretched along lateral axis, because the bandwidth is measured by its height of the graph mapped on the field. That is, the maximum length of a graph is an important factor for the numerical ordering of nodes.

For tree graphs, a number of nodes have more than three degrees and they are called "centres", though the degree of the other nodes is equal to one or two.

A node with deg. = 2 is connected to two nodes by d = 1. Thus, if a number of nodes with deg. = 2 are connected each other, they form a nodal sequence in which they are connected in series. In the sense, a tree graph is the one in which a number of nodal sequences are gathered at centres.

For a tree graph, the interchanges of the ordering of nodal sequences around a centre don't alter the topology.

In this chapter, the author proposes a new reduction method of bandwidth for tree graphs by use of the two-dimensional filing field in the previous chapter. There, we treat a tree graph as to be a gathering of nodal sequences and by using the method, the mapped graph in the field may have the narrowest width (i.e. the height in the field).

5–2. Preparatory Works for Sequential File Method

A linear graph, G, is a configuration which is drawn only by nodes and lines. Joints and members of a framed system are presented by nodes and lines, respectively.

In graph theory, a group of some divided subgraphs are also treated as a graph, but in structural analysis they are treated as some independent structures. Thus, a structure is drawn by a connected graph.

As the problem in consideraion is the one of bandwidth of stiffness matrix, some lines corresponding to members which are connected to datum node (i.e. to the ground) may be removed from the graph and they are ignored for the bandwidth problem. Any line can connect only two nodes and some of them connect nodes to ground. Thus, their influence for K matrix in previous chapter appears on the diagonal elements and not on the off-diagonals. The bandwidth problem is the rearrangements of non-zero elements which appear in off-diagonals. Therefore, we may remove them all from the graph for bandwidth problem. In the sense, the graph for bandwidth problem is a modification of the graph which is used for usual analysis of network problem. The difference is shown in Fig. 5–1–a, b and c. Fig. 5–1–a presents an actual framed structure, and Fig. 5–1–b shows the graph for usual network analysis which includes three branches connected to datum node. They are removed from the graph and we obtain a graph for bandwidth problem, as shown in Fig. 5–1–c. Therefore, if an original frame has m_d members which are connected to the datum , we have following relation between original system and its graph;

$$\text{Original System } (n, m + m_d) \rightarrow G(n, m)$$

G(n, m) means a graph with n nodes and m lines.

The removal of branches can be done when the datum node has no sense except the supporting joint, and when it shows no displacement during the loading. But, if the datum node displaces after the loading, we have to take the node into consideration for the analysis and we give nodal numbering for it. In the case, we cann't remove the node from the graph for nodal labeling and, also, we cann't put the members away. Therefore, we obtain the same graph for the nodal labeling as the one for its analysis. That is, the configuration shown in Fig. 5-1-b is treated for numerical ordering of nodes.

The description in above sentences is kept for any structural system.

5–3. Sequential File Method for Tree Graph

In accordance with the considerations for bandwidth which are done in this chapter, the author proposes a reduction method of bandwidth of stiffness matrix for tree system.

The most important role is played by

1. the diameter and
2. the maximum degree

of tree graphs.

The method proposed here is a graphical one which removes the complexity of the connectivity relationship of graph.

5–3–1. In the case of tree graph with one centre ($d_0 = 2$ and $r_0 = 1$)

An example of this tree graph is presented in Fig. 3-5-a. If it includes n nodes, the maximum degree of the centre is (n-1). In this case, it is evident that the bandwidth can be reduced to minimum value when the numerical number of the centre is ordered to be equal to the middle one of the numbers. Thus, H. B. W. is obtained by following equation.

$$\text{H. B. W.} = m - [m/2] + 1 \tag{5-1}$$

, where m indicates the maximum degree of the graph.

Followingly, we consider how to represent H. B. W. by graphical method. According to the above considerations, the number of nodes included between the first and the centre nodes should be equal to the rest. Using this suggestion, we can give a graphical representation to minimize H. B. W. as following steps. The procedures can be easily understood by the reference of Fig. 5-2.

(1). Selecting arbitrary two nodes except the centre.

(2) These two nodes with the centre compose the diameter and they are arranged on a lateral thick line with length 2 in the figure. This line shows the diameter, d_0.

(3) As three nodes among 7 in the example are already chosen, the rest have to be placed in the new graph. Two of them are placed above the centre and the rest above the node which is directly connected to the centre by d = 1 and placed on the left side of the centre. Thus, they are rearranged in two columns with same height, i.e. 2. If there are x nodes as the rest, the height of two columns are [x/2] and x

$\lfloor x/2 \rfloor$.

(4). The four nodes rearranged in the previous step are newly connected to the centre. The new graph is different from the original one at a glance, but they are topologically the same one and have the same connectivity relationship.

(5). Labeling of number for nodes is performed in accordance with the arrow presented in Fig. 5-2. That is, it is done from the right side to the left and also from top to bottom. By this operation, the center is ordered as it is wanted.

In this graphical method, the length and the width of a tree graph are represented in lateral and vertical directions, respectively. For this example, the maximum length and width are governed by the diameter and the degree of the centre, respectively. Furthermore, H. B. W. of the graph is obtained as the maximum width of the rearranged graph.

By use of the method, the maximum difference value of two node-numbers which is concerned to H. B. W. is found between two nodes connected by a lateral line at the location with maximum rows. In above example, it is found between two nodes on the diameter.

Furthermore, H. B. W. can be calculated only by counting the maximum width of the graph (i.e. the number of rows) and using the equation;

$$\text{H. B. W.} = (\text{H. B. W. of diameter}) + (\text{number of rows except diameter}) \quad (5-2)$$

H. B. W. of any diameter is always equal to 2. Thus,

$$\text{H. B. W.} = 1 + (\text{number of rows}) \quad (5-3)$$

In the example, maximum width = 2 and H. B. W. of the graph is obtained to be equal to 4. This coincides with the fact.

Fig. 5-2 explains the graphical representation of a tree graph with one centre and $(2n+1)$ nodes. The degree of the centre is equal to $2n$ and it is obvious that n rows including the diameter are necessary to draw a rearranged graph. We notice from this fact that H. B. W. will increase at least by one, when the degree of the centre increases by two.

The allowable directions of a connecting line are restricted to be lateral, vertical, and obliquely descending to the right or ascending to the left as shown in Fig. 5-2. Thus, every two nodes in neighbouring columns can be connected in the range of allowable directions and two nodes in the same column can be connected without restrictions.

5-3-2. In the case of a general tree graph with one centre

Above example in Section 5-3-1 is restricted to the case of $d_0 = 2$ and $r_0 = 1$. In this section, we treat the one whose nodal sequences can have arbitrary length.

Consider a centre having m degree. In order to obtain the diameter, we compare m nodal sequences gathering at the centre and select the longest two sequences which include d_1 and d_2 nodes except the centre, respectively. These two sequences with the centre compose the diametre, namely, $d_0 = d_1 + d_2$.

By the application of the graphical representation to this case, the general tree with one centre can be drawn as shown in Fig. 5–3. An arbitrary nodal sequence can be stored in a row along the diameter. Thus, it concludes that H. B. W. of a general tree graph is decided by the number of nodes which locate at $d = 1$ from the centre, in other words, by the degree of the centre. Therefore, this case is just the same as the case of a tree with $d_0 = 2$ and $r_0 = 1$.

5–3–3. In the case of a tree with two centres

An example of a tree graph with two centres is shown in Fig. 5–4. As the graph is a tree, there exists only one line between two centres and its length is denoted by d_2.

The degrees of two centres are denoted by m_2 and m_3, and the longest sequences among m_2 and m_3 sequences are selected except d_2 and are denoted by d_1 and d_3, respectively. The nodal sequence, $(d_1 + d_2 + d_3)$, is selected as a temporary diameter of the graph.

In Fig. 5–4, 1 and 4 present the end nodes of the diameter, and 2 and 3 are the centres. d_2^i ($i = 1, 2, \ldots, m_2 - 2$) and d_3^j ($j = 1, 2, \ldots, m_3 - 2$) present the length of nodal sequences from 2 and 3 nodes, respectively. d_1, d_2 and d_3 are not included among them.

The word "temporary diameter" is used, because it has a posibility that the nodal sequence, $(d_1 + d_2 + d_3)$, does not compose the diameter, when $d_2^1 > d_2 + d_3$ or $d_3^1 > d_1 + d_2$. The author gives considerations about these cases at the end of this section. At this stage, the sequence, $(d_1 + d_2 + d_3)$, is supposed to be the longest one and it is treated as a true diameter.

$$d_0 = d_1 + d_2 + d_3 \tag{5–4}$$

and, we have following relations between nodal sequences.

$$\begin{aligned} d_1 &\geqslant d_2^i \geqslant d_2^{i+1} \qquad (i = 1, 2, \ldots, m_2 - 2) \\ d_3 &\geqslant d_3^j \geqslant d_3^{j+1} \qquad (j = 1, 2, \ldots, m_3 - 2) \end{aligned} \tag{5–5}$$

In the case of a tree graph with one centre, H. B. W. is decided only by the maximum degree, but in the case with two centres the problem is not so simple but much more complicated and we have to introduce another concept which is called "the nodal capacity".

When all of the nodes included in the tree graph in Fig. 5–4 is optimally labeled, all of the sequences are filed within h rows except the diameter by use of the sequential file mehtod, The value, h, has to satisfy following relations ;

$$h \geqslant m_2 - [m_2/2] - 1$$

$$h \geqslant m_3 - [m_3/2] - 1 \qquad (5-6)$$

At this stage, we call the number of nodes, which can be stored between nodes 1 and 2, 2 and 3, and nodes 3 and 4, the nodal capacities and they are described by Cap. $1 \sim 2$, Cap. $2 \sim 3$ and Cap. $3 \sim 4$, respectively.

$$\begin{aligned} \text{Cap. } 1 \sim 2 &= h \cdot d_1 \\ \text{Cap. } 2 \sim 3 &= h \cdot d_2 \\ \text{Cap. } 3 \sim 4 &= h \cdot d_3 \end{aligned} \qquad (5-7)$$

Any nodal sequence, which is placed in the left side of 2-node and also in the right side of 3-node, can be filed in a row, but a sequence which should be filed between two centres may occupy more than two rows, if its length is longer than d_2. Thus, it is obvious that the region between two centres has to keep enough rows in order to place the nodes which are connected to the centre by $d = 1$. Furthermore, the area must be larger than the total number of nodes which belong to the sequences placed in the central region. This can be presented by following equation.

$$\text{Cap. } 2 \sim 3 \geqslant \sum_i d_2^i + \sum_j d_3^j \qquad (5-8)$$

This equation gives the suggestion that the sequences to be placed in the central region should be selected successively from the shortest one among the sequences.

Above considerations show that the minimum bandwidth of tree graphs with two centres can be obtained not only to consider the maximum degree of a centre but also to pay attentions to the length of sequences and the nodal capacity between two centres.

Followingly, the author proposes some graphical steps to obtain the minimum width of a tree graph with two centres. The steps are for the general tree graph as shown in Fig. 5–4.

[Step–1]. Selection of a temporary diameter of the graph.

The temporary diameter is selected as shown at the beginning of this section. This diameter is drawn by a straight line in Fig. 5–5.

[Step–2]. Calculation of the initial width of the tree graph.

After Step–1, we consider only the nodes which are connected to the centres by $d = 1$ and take enough width to place them in the new graph. Thus, the method proposed for the case of one centre is directly applied for Step–2. Then, we calculate the initial width (i.e. the number of rows) above the centres, 2 and 3 nodes, and they are described by h_2 and h_3, respectively. They are calculated as followings.

$$\begin{aligned} h_2 &= m_2 - [m_2/2] - 1 \\ h_3 &= m_3 - [m_3/2] - 1 \end{aligned} \qquad (5-9)$$

These values are drawn by vertical lines above the centres and also above the nodes locating on the left side of them by $d = 1$. (See Fig. 5-5)

[Step-3]. Selection of nodal sequences to be located on the left side and the right side of the centres, 2 and 3 nodes, respectively.

According to the consideration which is done in this section, they should be selected successively from the longest nodal sequence among ($m_2 - 2$) and ($m_3 - 2$), respectively. After the selection of h_2 and h_3 sequences, they are filed in rows along the diameter. Their initial nodes (i.e. the nodes with $d = 1$ from the centres) are placed in columns which locate above the node neighbouring to the centre and above the centre respectively. The state after this step is presented in Fig. 5-5.

At this step, nodal sequences, ($d_2^1, d_2^2, \ldots, d_2^{h_2}$) and ($d_3^1, d_3^2, \ldots, d_3^{h_3}$), are filed and the rest sequences are left for following steps.

[Step-4]. Filing operation of $|h_2 - h_3|$ nodal sequences.

At this step 4, we select $|h_2 - h_3|$ sequences to be placed between ($h_3 + 1$) and h_2 rows, if $h_2 \geqslant h_3$. By a proper selection of $|h_2 - h_3|$ nodal sequences, we can leave the least number of nodes belonging to the sequences which are not filed at this step and will be filed in Cap. $2 \sim 3$ at the following step.

They are selected successively from the longest to the $|h_2 - h_3|$-th sequence by use of the comparison table of residual nodal sequences which is newly introduced and is shown in Fig. 5-6. As h_2 is larger than or equal to h_3 in this example, the selected sequences are filed as shown in Fig. 5-7, that is, the initial nodes which are distant from the centres by $d = 1$ are ordered in two columns. The distance between two columns coincides with the distance of two centres of the original tree graph, namely d_2 for this example.

After the step-4, there leave enough unoccupied points in two columns above 2-node and next to 3-node for the initial nodes of unfiled nodal sequences being left for step-5.

[Step-5]. Checking whether the residual nodal sequences can be filed in Cap. $2 \sim 3$.

In order to file the residual sequences at step 4 in the area between two centres, it is necessary that the nodal capacity of the area is, at least, equal to or larger than the summation of the nodes belonging to the sequences. If the summation of nodes is denoted by S, it can be expressed by following relation.

$$\text{Cap. } 2 \sim 3 \geqslant S = \left(\sum_{i=\alpha}^{m_2-2} d_2^i + \sum_{j=\beta}^{m_3-2} d_3^j \right) \tag{5-10}$$

$$\left|\begin{array}{l} \alpha = m_2 - h_2 - \Delta h_2 - 1 \\ \beta = m_3 - h_3 - \Delta h_3 - 1 \end{array}\right.$$

If Cap. $2 \sim 3 < S$, it is obvious that the nodal capacity between $2 \sim 3$ is insufficient to file the sequences. In this case, the operation for sequential filing has to be returned to the previous step. Then, by use of Fig. 5-6, even number of nodal sequences are newly selected and filed above the h_2-th row, until the relation (5-10) is satisfied. If

$2\Delta h$ sequences are newly selected and filed additionally, the total rows becomes ($h_2 + \Delta h$), and this state is shown in Fig. 5 7.

If the assumption that the nodal sequences, d_2^i and d_3^j, can be filed in Cap. $2 \sim 3$ without leaving unoccupied area is right, the minimum width of a filed graph can be decided by comparing S with Cap. $2 \sim 3$ and by operating the additional filing until the establishment of the relation (5–10). Thus, the half bandwidth is obtained by following equation.

$$\text{H. B. W.} = 2 + h \tag{5-11}$$

Here, the author shows the rightness of the above assumption, The area of Cap. $2 \sim 3$ is shown in Fig. 5–8. The critical case of the assumption is that the summation of nodes which belong to the residual sequences is equal to the nodal capacity. It is expressed by the equation,

$$\text{Cap. } 2 \sim 3 = n \cdot d_2 = \bar{S}_2 + \bar{S}_3 = \sum_i d_2^i + \sum_j d_3^j \tag{5-12}$$

, where $\bar{S}_2$ and S_3 are total nodes belonging to d_2^i and d_3^j, respectively.

The area of Cap. $2 \sim 3$ is divided into two subareas, $\bar{S}_2$ and $\bar{S}_3$ as shown in Fig. 5–8, in accordance with following operation; At first, the area is divided into two subareas at the $[\bar{S}_2/n]$-th column from the left centre and by the addition of ($\bar{S}_2 - [\bar{S}_2/n] \times n$) nodes in the ($[\bar{S}_2/n] + 1$)-th column to the subarea on the left side, we can divide S_2 area in Cap. $2 \sim 3$. Thus, the residual area is obviously equal to $\bar{S}_3$. In order to file d_2^i in $\bar{S}_2$, we begin to file from the shortest one and sufficient rows are used for a sequence in order to avoide the reflection at the boarder line between $\bar{S}_2$ and $\bar{S}_3$, as shown in Fig. 5–8. This operation is repeated for all d_2^i and $\bar{S}_2$ is covered by d_2^i using the allowable directions of connecting lines. This procedure is repeated to file d_3^j in the area of $\bar{S}_3$. Thus, the area of Cap. $2 \sim 3$ is just covered by the nodal sequences, if Cap. $2 \sim 3$ $= \bar{S}_2 + \bar{S}_3$.

By use of the rightness of the above assumption, the minimum H. B. W. of a tree with two centres is graphically obtained in accordance with the five steps described in this section and H. B. W. is given by eq. (5–11). The value, h, is the width of the tree graph.

In the beginning of this section, the author described about the exceptional cases where ($d_1 + d_2 + d_3$) sequence does not compose the true diameter. Here is given further considerations about it. The exceptional case happens when two nodal sequences from one centre compose the diameter. If the degree of the centre with two nodal sequences, which compose the diameter, is less than that of another centre, it is obvious from Fig. 5–9 that the direct application of the sequencial file method to the case has the tendency not to give the accurate result for H. B. W. but to give the result which is bigger than the minimum H. B. W. by one.

Thus, it concludes that the application of the temporary diameter instead of the true one gives the accurate result for the sequencial file method in the case of two centres.

5-3-4. In the case of a tree graph with three centres.

In this section, the author explains the sequential file method for the tree graph with three centres and it is just similar to the one which is described in the previous section.

An example of general tree graph is shown in Fig. 5-10. The centres are described by 2, 3 and 4-node, and the degree of them are ($m_2 + 2$), ($m_3 + 2$) and ($m_4 + 2$), respectively.

Between every two centres, there is only one nodal sequence and the longest sequences among ($m_2 + 1$) and ($m_4 + 1$) are selected. These four sequences compose the temporary diameter of the graph, as shown in Fig. 5-10.

$$d_0 = d_1 + d_2 + d_3 + d_4 \qquad (5\text{-}13)$$

The example is the one with $m_3 \geqslant m_2$ and $m_3 \geqslant m_4$. Followingly, the procedures for the sequencial file method are described briefly.

[Step-1]. Selection of temporary diameter of the graph.

d_0 is shown by a lateral straight line in Fig. 5-11.

[Step-2]. Calculation of the initial width of the graph.

For the reservation of enough rows which are necessary for the degree of the centres, following calculations are done.

$$\begin{aligned} h_2 &= m_2 - [\, m_2/2 \,] \\ h_3 &= m_3 - [\, m_3/2 \,] \\ h_4 &= m_4 - [\, m_4/2 \,] \end{aligned} \qquad (5\text{-}14)$$

h_2, h_3 and h_4 rows are reserved above the three centres, respectively and they are shown by the vertical lines in Fig. 5-11.

[Step-3]. Selection of nodal sequences to be located on the left and the right side of 2- and 4-node, respectively.

Among m_2 sequences, the longest h_2 sequences are selected and filed on the left side of 2-node. The same procedure is done for the area on the right side of 4-node. The state after this step is shown in Fig. 5-11.

[Step-4]. Selection and filing procedure of ($h_3 - h_2$) and ($h_3 - h_4$) sequences.

At the state after this step, we have to leave minimum number of nodes in order to file the central area between 2 and 4 nodes. That is, we select the longest sequences at this step. For this purpose, we use the comparison table of residual nodal sequences as shown in Fig. 5-12 and select ($h_3 - h_2$) and ($h_3 - h_4$) sequences among (I) and (II), and (I) and (III) group, respectively. The sequences selected by the above procedure are filed above the graph shown in Fig. 5-11 and the state after this step is presented in Fig. 5-13. The method to make the comparison table is given in previous section. Δh_2 among ($h_3 - h_2$) are selected from (II) group, and Δh_4 among ($h_3 - h_4$) are from (III) group.

[Step-5]. Checking whether the residual sequences can be filed in Cap. $2 \sim 4$.

The nodal capacity between every two centres are fixed by the previous step and they are given by the following equations.

$$\begin{aligned} \text{Cap. } 2 \sim 3 &= (h_2 + \Delta h_2) \cdot d_2 \\ \text{Cap. } 3 \sim 4 &= (h_4 + \Delta h_4) \cdot d_3 \end{aligned} \qquad (5-15)$$

and

$$\text{Cap. } 2 \sim 4 = \text{Cap. } 2 \sim 3 + \text{Cap. } 3 \sim 4 \qquad (5-16)$$

The total number of nodes belonging to unfiled nodal sequences is counted for every groups (I), (II) and (III), and they are described by $\bar{S}_2$, $\bar{S}_3$ and $\bar{S}_4$, respectively.

$$\begin{aligned} \bar{S}_2 &= \sum_{i=h_2+\Delta h_2+1}^{m_2} d_2^i \\ \bar{S}_3 &= \sum_{j=\gamma}^{m_3} d_3^j \\ \bar{S}_4 &= \sum_{k=h_4+\Delta h_4+1}^{m_4} d_4^k \end{aligned} \qquad (5-17)$$

, where $\gamma = 2h_3 \; h_2 - h_4 - \Delta h_2 - \Delta h_4 + 1$.

Thus, $S = S_2 + \bar{S}_3 + \bar{S}_4$ gives the total nodes of unfiled sequences.

If $\qquad S > \text{Cap. } 2 \sim 4, \qquad (5-18)$

the nodal capacity is obviously insufficient for the filing of the residual sequences. Then, the step 4 is applied again, till the equation

$$S \leqslant \text{Cap. } 2 \sim 4 \qquad (5-19)$$

is established by selecting the additional sequences and by filing them over h_3 rows. This procedure is explained in the previous section and is done by use of the comparison table of sequences. The state where eq. (5-19) is established is presented in Fig. 5-13.

At this stage, Cap. $2 \sim 4$ has sufficient nodal capacity for the unfiled sequences. But the relations between S_2 and Cap. $2 \sim 3$, and $\bar{S}_4$ and Cap. $3 \sim 4$ are not related. Thus, we have following three cases;

$$\text{Case 1;} \quad \begin{aligned} &\bar{S}_2 \leqslant \text{Cap. } 2 \sim 3 \quad \text{and} \\ &\bar{S}_4 \leqslant \text{Cap. } 3 \sim 4 \end{aligned} \qquad (5-20)$$

$$\text{Case 2;} \quad \begin{aligned} &\bar{S}_2 > \text{Cap. } 2 \sim 3 \quad \text{and} \\ &\bar{S}_3 + \bar{S}_4 < \text{Cap. } 2 \sim 4 - \bar{S}_2 \end{aligned} \qquad (5-21)$$

$$\text{Case 3;} \quad \begin{aligned} &\bar{S}_4 > \text{Cap. } 3 \sim 4 \quad \text{and} \\ &\bar{S}_2 + \bar{S}_3 < \text{Cap. } 2 \sim 4 - \bar{S}_4 \end{aligned} \qquad (5-22)$$

They are shown in Fig. 5-14. These cases are treated by following procedures.

[The treatment for case 1]

If the length, b, is longer than a, the area (Cap. 3 ~ 4 $\bar{S}_4$) can store as many sequences among $\bar{S}_2$ as the number of rows of the area. Thus, we select them from the longest one successively and file them in the area. At this state, a number of unoccupied points are left in the area, Cap. 3 ~ 4. Then the last operation for filing Cap. 2 ~ 3 is similar to the procedure for the case of two centres. The area, Cap. 2 ~ 3, has insufficient nodal capacity for the filing of $\bar{S}_2$ and the unfiled sequences of $\bar{S}_3$. Then, even number of additional sequences are selected among them by use of the comparison table and they are additionaly filed above the rows already filed in the previous steps, until the rest can be stored in the area.

[The treatment for case 2]

At first, $\bar{S}_4$ is filed in Cap. 3 ~ 4 and we select as many longest sequences as the number of rows of Cap. 3 ~ 4 among $\bar{S}_3$ and file them in the area. The procedure after this state is just the same as that of case. 1.

[The treatment for case 3]

In this case, the area of Cap. 2 ~ 3 is filed at first and the treatment after this is just the same as the previous cases.

By use of these five steps in the case of tree graph with three centres, the original graph is modified into a new one with minimum width without changing the topological property. If it has n rows after the operation, we can conclude that the minimum half bandwidth of the tree graph can be reduced to (2 + n).

The above method is done by use of the temporary diameter. If the true diameter does not pass three centres but passes less than three centres and the degree of the centre which does not compose the diameter is larger than the others, H. B. W. obtained by use of the true diameter does not give the minimum value but increases by one. That is, the state appeared in the case of two centres can be found in this case, too.

Therefore, it can be said that we should use the temporary diameter for the application of the sequencial file method to tree graphs with more than two centres.

5–3–5. General tree graph with centres in series

The author already explained the sequntial file method applied to tree graphs with one, two and three centres.

In this section, he treats a tree graph with a number of centres in series and explains briefly the application of sequential file method to it.

As far as the centres are connected in series, the method can be directly applied to the graph, but the actual and strict application becomes as complicated as the number of centres increases.

The method proposed here is the one to obtain the minimum bandwidth of a tree graph, and as the number of centres increases, the number of comparison of length of sequences increases and it takes much time.

Now, the author describes the brief procedure of the application to tree graphs.

[Step-1]. Selection of diameter.

At first, a number of lines are selected to construct the diameter of the graph. The diameter may be a temporary one. If the graph has α centres, there exist only one path between every neighbouring two centres and they are selected as to compose $(\alpha - 1)$ elements of the diameter. Among lines which are connected to both ends of a series of centres, the longest lines are selected and all of them compose the diameter. This diameter, d_0, is presented by a thick line in Fig. 5–15.

[Step-2]. Calculation of the initial width of the graph.

For the reservation of enough rows which are necessary for the degree of centres, following calculations are done.

$$h_i = m_i - [m_i/2] \tag{5-23}$$

, where i is the number of centres. At this step, we obtain $h_1, h_2, h_3, \ldots, h_i, \ldots, h_\alpha$. Nodal rows corresponding to the values are reserved above the centres, respectively and they are drawn by the vartical lines in Fig. 5–15.

[Step-3]. Selection of nodal sequences to be located on the left side and the right side of centres, i.e. 1 and α, respectively.

The operation is just the same as described in previous sections of this chapter. The state of field after this step-3 is shown in the same figure.

[Step-4]. Selection and filing procedure for intermediate area and also the side-area of end-centres.

For every filing area between neighbouring centres where nodal sequences are not yet filed, all of the nodal sequences are compared and selected as many sequences as the number of unoccupied rows for every centres. For this purpose, we use the comparison tables which are illustrated in previous sections. In accordance with the procedure described already, the filing field is occupied by selected nodal sequences as many nodal rows as the maximum number of h_i which is already illustrated in Fig. 5–16. The state after this step is presented in Fig.5–16.

[Step-5]. Checking whether the unfiled sequences can be filed in Cap. $1 \sim \alpha$.

At step-4, a number of nodal sequences from every centre are selected from the longest one and filed as shown in Fig. 5–16. In the figure, we find Cap. $1 \sim \alpha$ leaving unoccupied and we calculate Cap. $1 \sim \alpha$. At the same time, the total number of nodes which are included in unfiled sequences is calculated and is denoted by S.

If

$$S \leqslant \text{Cap. } 1 \sim \alpha, \tag{5-24}$$

it may be possible that all of the unfiled nodal sequences are filed in Cap. $1 \sim \alpha$.

If

$$S > \text{Cap. } 1 \sim \alpha, \tag{5-25}$$

we have no possibility that they are filed under the h_i-th row. In general, we need not operate this checking procedure for all of the field, but for only a part of it. It is caused by the reason that a nodal sequence is short enough to cover only a few centres. Thus, if the distance between two neighbouring centres is relatively short and their degrees are large and also the nodal sequences from them are long comparing to the length between the two centres, this area in the field may be critical for filing. Therefore, we may check only this area.

If the second case is obtained, we have to repeat to select some more nodal sequences and to file them above the h_i-th row, till the equation for the first case is established. For this repetion of the procedure we use the comparison table for residual nodal sequences and the procedure may be shown in the calculation table.

After the first equation is established, the total area of the nodal capacities between every neighbouring centres prepares enough nodal capacity for the unfiled nodal sequences. Then, we continue to file redundant sequences in the area by the procedures which are given in previous section.

As described in this section, more centres a tree contains, more complicated the procedure becomes. But, as far as the centres are included in series, it is general that the critical part of it for the minimization procedure of H. B. W. is restricted only in a part of it. The part may be easily found only by the inspection and observation of the whole graph. The other part of the graph has no influence for the minimization of H. B. W. Thus, if we treat a graph, we should, at first, observe the whole system and seek out the part which will be critical and govern the value of H. B. W..

As far as we treat an actual tree structure, the graph is simpler one than those which the author showed in these sections. In actual and general tree structure, nodes are located at the points where more than two members are connected or where loads are concentrated on. And a general nodal sequence has not so many nodes on it as an nodal sequence in an abstract graph can contain. At the same time, the same fact can be said for a nodal sequence between two centres.

These characteristics of tree graphs corresponding to actual tree structures may, in general, become very effective for actual application of the above mentioned method to obtain the minimum value of bandwidth. And only a part of the system can be influent on and relate to the minimization procedure of the half bandwidth. Or, even if whole system may concern with the minimization, the application of the proposed method is not troublesome, because any centre gathers not so many nodal sequences as an abstract one can have, and also any nodal sequence contains only a few nodes.

5-3-6. Tree graphs with centres located not in series

In this section, the author treats a tree graph which includes a number of centres that are connected each other not in series.

As far as the number of centres is less than or equal to three, they are necessarily connected in series. Their treatments and how to minimize H. B. W. are already described. But if the number is more than or equal to four, they may be located not in series. That is, a centre is connected to more than two centres by nodal sequences, though for a tree with centres in series every centre is connected to neighbouring one or, at most, two centres by nodal sequences.

Fig. 5 17 shows the simplest example among the above mentioned tree graphs. It includes only four centres and one of them is the centre of the other three centres. Using this example, the author explains how to obtain the numerical ordering of nodes which gives the minimum bandwidth.

If one of three centres, i.e. C_1, C_3 or C_4, is removed from the graph, the residual tree is the one with three centres, and it is already familar for us. By the addition of the fourth centre, the system is changed as a whole. But, as far as we treat the graph as a gathering of nodal sequences and also we use the filing field in two dimensional space, our object is how to file the graph in the field as to have the minimum height.

Followingly, the author explains how to obtain the minimum bandwidth for this type of tree graphs and at the end of this section he discusses the demerit of the method proposed here.

The characteristics of the tree graph are given in Table. 5 1.

[Step 1]. Selection of the diameter of the graph.

The relation, i.e. $d_1 \geqslant d_2 > d_3$, is assumed for the tree graph in Fig. 5 17. In order to use the two-dimensional filing field we have to investigate the diameter which is the base for filing nodal sequences. For the first step, we select every two centres and centre of centres. And by paying attention to the location of the fourth centre, we obtain three cases for the selection of diameter, and these three cases are shown in Fig. 5 18 a, b, and c. That is, the normal case for selecting the diameter is presented in Fig. 5-18 a and b, because the two nodal sequences, e.g. d_1 and d_2 which connect C_1, C_2 and C_3, are the longest two sequences and Cap. $C_1 \sim C_3$, i.e. the nodal capacity between the two centres, may reserve the largest number of unoccupied space for nodes. Thus, the general cases of the diameter correspond to a and b in Fig. 5–18, respectively, and the third case is treated after the description about the procedure for these two cases.

Among nodal sequences from C_1 the longest sequence is selected. This procedure is repeated for C_3. From Table 5-1 we obtain 1_1^1 and 1_1^3 for them. The diameter, d_0, yields to be

$$d_0 = 1_1^1 + d_1 + d_2 + 1_1^3 \qquad (5\text{-}26)$$

As described in Section 5 3-3, the value obtained by the equation may not be the longest value and it is, of course, the temporary diameter. The diameter is presented by a thick line in Fig. 5–19.

[Step–2]. Optimization for graph with three centres, i.e. C_1, C_2 and C_3.

This step coincides with the optimization for a tree graph with three centres which was described in previous section. Using the diameter, d_0, the new graph which is obtained by removing the fourth centre, C_4, and is necessarily the one with three centres, is filed in the two-dimensional filing field by the procedure which is proposed in previous sections. At the optimized stage, we count the height of the mapped graph in the field and the value is denoted by $\overline{H.B.W}$..

The final configuration of mapped graph should be concave between C_1 and C_2 or C_2 and C_3, as shown in Fig. 5–19. To obtain the configuration, the procedure for optimization up to Step 2 should be strictly followed as described in previous section and the value, ΔS, in the calculation table must be positive, though the last stage before becomming positive may also show a kind of optimum state in which the central part of the graph is convex.

The concave is the preparation for the next step of filing the nodal sequences from the fourth centre above the result obtained at Step 2. At this stage, it becomes obvious that the method proposed here for a tree graph with four centres is just a modification of the method for three centres and we use the result for three centres. That is, we obtain the minimized half bandwidth for the case of four centres by additional filing of nodal sequences from the fourth centres above the results.

Therefore, we may have misgiving that the optimization for four centres cann't be followed from the result for three centres. But the fear may be put away by following reason.

$d_1 \geqslant d_2 > d_3$ is assumed. Then, the area where the centre, C_4, can be exist is, of course, between C_1 and C_3, and it cann't be reached at C_1 or C_3. From this fact it is obvious that half of the nodal sequences from C_1 and C_3 are, at least, filed on the left side of C_1 and also on the right side of C_3. This fact shows that the filing technique for the case of three centres can be applicable for this case of four centres, and we may consider the influence of the appearance of the fourth centre only for the filing after the filing step described here. That is, the nodal sequences from C_4 can effect only for the filing between the two centres, i.e. C_1 and C_3, and the additional filing above the occupied area of the outer side of C_1 and C_3. Furthermore, by the direct application of the method for three centres, we may have the least value of the half bandwidth of the case with four centres, because the case has to include more nodes than the case with C_1, C_2 and C_3. Thus, we obtain the relation;

$$H.B.W. \geqslant \overline{H.B.W.} \tag{5-27}$$

By the reason, we find, at least, the half bandwidth of the graph except the C_4 centre.

[Step-3]. Simple filing of C_4-sequences above $\overline{H.B.W}$.

This step is to observe the maximum half bandwidth which is obtained by simple filing of nodal sequences from the fourth centre above the result obtained by the step 2.

The centre, C_4, may be located between $C_1 \sim C_2$ or $C_2 \sim C_3$. For example we take, here, the first case which is visualized in Fig. 5–18–a. By the additional filing of nodal sequences from C_4, they may have, at most, following height which is decided only by the degree of C_4.

$$\Delta(\mathrm{H.B.W.}) = m_4 + 1 - \left[\frac{m_4+1}{2}\right] \tag{5-28}$$

, where 1 shows the sequence from C_2 to C_4, and $\Delta(\mathrm{H.B.W.})$ is the additional height.

If the bottom of the convex area has the height, $\overline{\overline{\mathrm{H.B.W.}}}$, the bandwidth by the simple filing, $\widetilde{\mathrm{H.B.W.}}$, may be expressed by following inequality :

$$\overline{\mathrm{H.B.W.}} + \Delta(\mathrm{H.B.W.}) \geqslant \widetilde{\mathrm{H.B.W.}} \geqslant \overline{\overline{\mathrm{H.B.W.}}} + \Delta(\mathrm{H.B.W.}) \tag{5-29}$$

The value, $\widetilde{\mathrm{H.B.W.}}$, gives the maximum value for the graph with four centres, and it is related to the true half bndwidth, H. B. W. of the graph by following inequality: (See Fig. 5-20)

$$\widetilde{\mathrm{H.B.W.}} \geqslant \mathrm{H.B.W.} \tag{5-30}$$

The value is to be reduced to the true one by following procedures.

If the location of the concave is out of d_3 from C_2, the value $\overline{\overline{\mathrm{H.B.W}}}$. is meaningless, and we treat the value $\overline{\mathrm{H.B.W}}$. for the investigation of $\widetilde{\mathrm{H.B.W.}}$. The reason is that the nodal sequences from C_4 have their termini at C_4 which is located within d_3 from C_2.

[Step 4]. Filing procedure of nodal sequences from C_4 up to $\overline{\mathrm{H.B.W}}$.

At first, the sequences which are filed between C_1 and C_2 are newly refiled there in order to secure the maximum number of unoccupied nodal rows around d_3 from C_2. For the purpose all the sequences are newly and successively filed from the shortest one. This procedure is the preparatory one for reserving unoccupied space for C_4-sequences and by the operation the bottom of the concave is made as deep as possible by the reordering of already filed nodal sequences.

Followingly, C_4-sequences are filed there from the shortest sequence, because the degree of C_4 governs the half bandwidth of the graph with C_4 and the vacant space between C_1 and C_2 should be occupied by as many number of nodal sequences as possible in order to leave the least number of nodal sequences from C_4 which are to be filed at next step.

If all of the C_4-sequences can be filed there, we can conclude that the half bandwidth of the graph with four centres is equal to $\overline{\mathrm{H.B.W.}}$.

$$\mathrm{H.B.W.} = \overline{\mathrm{H.B.W.}} \tag{5-31}$$

In general, only a number of sequences can be filed there and the residuals have to be filed over $\overline{\mathrm{H.B.W.}}$. If there leave m_α sequences unfiled, the true half bandwidth may be related by following relation.

$$\mathrm{H.B.W.} < \overline{\mathrm{H.B.W.}} + m_\alpha - \left[\frac{m_\alpha}{2}\right] \tag{5-32}$$

This relation is obtained by simple filing of the residual sequences above $\overline{H.B.W.}$, as shown in Fig. 5-21.

If less than two sequences are left after this step, we can easily obtain H. B. W. by following equation.

$$H.B.W. = \overline{H.B.W.} + 1 \tag{5-33}$$

[Step- 5]. Filing procedure of nodal sequences of C_4 above $\overline{H.B.W.}$

At this step, the residual sequences are filed above $\overline{H.B.W.}$ and we obtain the minimum half bandwidth for case-a in Fig. 5-18.

Assume that (m_4 - m_α) sequences among C_4-sequences are already filed at step 4. The shortest two sequences among the residuals are described by $l^4_{(m_4 - m_\alpha + 1)}$ and $l^4_{(m_4 - m_\alpha + 2)}$. We pick up the longest two nodal sequences among sequences which are from C_1 and C_2. Then, the four sequences are compared and the longest two are selected.

If the ones from C_1 and C_2 are selected, they are newly replaced and filed above the left side area of C_1 and the right side area of C_2, respectively. And the two from C_4 can be easily ordered between two centres, C_1 and C_2, because by removing the two sequences from C_1 and C_2, there is enough nodal space and also two degrees are reserved above the location of C_4.

On the other hand, if the sequence from C_1 is shorter than $l^4_{(m_4 - m_\alpha + 1)}$, we have to compare the latter with the shortest sequence from C_1 which is already filed within $\overline{H.B.W.}$ on the left side area of C_1, when it is more than the (m_1 - $[m_1/2]$)-th longest sequence, because for the left side of C_1 it is obvious that (m_1 - $[m_1/2]$) sequences from C_1 must be filed.

By the procedure given here, we can select four nodal sequences which are filed at the ($\overline{H.B.W.}$ + 1)-th nodal row in the field.

By the successive application of the procedure, every two residual nodal sequences are filed at additional one nodal row. When the two sequences from C_4 which are to be filed are shorter than the comparative sequences from C_1 and C_2 or there leave only (m_1 - $[m_1/2]$) and (m_2 - $[m_2/2]$) sequences on the left side of C_1 and on the right side of C_4, respectively, the comparison of sequences should be ended. At this stage, if there leave some nodal sequences unfiled in the field, they should be simply filed over the area. The final state of the filing sequences is illustrated in Fig. 5-22.

The half bandwidth of the filed tree graph is denoted by $H.B.W._{a}$ in which the subscript, a, denotes that the centre, C_4, is placed on the left side of C_2.

For the selection of the nodal sequence in above procedure, we have to use the concept of the comparison table. But for the strict application of the table, we must fix the location of C_4. That is, if we want to obtain $H.B.W._{a}$ strictly, we should calculate $H.B.W._{i}$ (i = 1, 2, 3, . . . , d_3) for every case in which (i) is fixed and we should obtain H. B. W. by comparing them and by selecting the minimum value among them. This is just the strict application of the proposed procedure and it leads us to the true optimum state of numerical ordering of nodes.

Actually, we need not calculate the every case for $i = 1, 2, 3, \ldots, d_3$. By the re-arrangement of nodal sequences between C_1 and C_2 and also by the removal of some nodal sequences from Cap. $C_1 \sim C_2$, the location of C_4 becomes evident from the filed graph, itself. Or, evern if the location is vague at the first glance of the graph, we can guess it by some preparatory operation of filing a few sequences.

The strict procedure gives following equation for $H.B.W._{\cdot a}$

$$H.B.W._{\cdot a} = \min. \text{ of } (H.B.W._{\cdot i}), \quad i = 1, 2, 3, \ldots, d_3 \tag{5-34}$$

[Step-6]. Investigation of H. B. W. for the case-b.

The case-b is presented in Fig. 5-18-b. The fourth centre C_4, is located on the right side of C_2.

In the investigation of H. B. W. for this case, the procedure from Step-1 to Step-5 is just applied and we obtain the minimum value of H. B. W. for the case and it is denoted by $H.B.W._{\cdot b}$.

$$H.B.W._{\cdot b} = \min. \text{ of } (H.B.W._{\cdot i}), \quad i = 1, 2, 3, \ldots, d_3. \tag{5-35}$$

, in which $H.B.W._{\cdot i}$ is the value for the centre located at $d = i$ from C_2.

In general cases whether the centre should be placed on the right or the left side of C_2 can be easily guessed by taking account of following items and the repetitive application of the steps given above may be removed for actual cases. The items are following two.

i) The longer the distance between two centres is, much capacity can be expected. Therefore, C_4 should be placed on the side with longer distance to neighbouring centres, i.e. d_1 or d_2.

ii) Comparing the nodal sequences which are already filed in Cap. $C_1 \sim C_2$ and Cap. $C_2 \sim C_3$, we should select the one which includes as many and also as longer nodal sequences in it, because by removing and reordering them we can reserve more unoccupied space for filing the sequences of C_4. If the above items don't give the answer, the analyst should preparatorily try to file a few C_4-sequences above result for three centres. And it will lead to the decision of the selection for the location of C_4.

From step-1 to step-6, the author doesn't treat the case, i.e. $d_1 \geqslant d_2 \geqslant d_3$. Here, he gives some investigation for the case. If the equalities are established for the graph, i.e. $d_1 = d_2 = d_3$, the centre, C_4, may be located just on the other centres. But this case is easily removed from the view of the reduction of half bandwidth.

On every centre, maximum height around the area must be prepared for filing sequences. Thus, the ordering of two centres in the same nodal column in the two-dimensional filing field should be taken off except some special cases, for example, the distance between two centres are equal to one.

[Step-7]. Investigation of H. B. W. for the case-c.

The case-c is presented in Fig. 5-18-c. In this case, the diameter includes the three centres, C_1, C_2 and C_4. And the fourth centre, C_3, is placed between C_1 and C_2. This case is the last one which is left in former steps.

In general, this case does not give the minimum value of H. B. W., because the temporary diameter for this case is shorter than the other cases. But, when the degree of C_4 is comparatively large enough, this case may give the critical value of bandwidth problem. In order to investigate H. B. W. for this case, the procedure proposed in this section is just followed and the result is denoted by $H. B. W._{\cdot c}$.

[Step 8]. Comparing the results of three cases.

The true half bandwidth of a tree graph with four centres is the minimum value among the results obtained in former steps.

Thus, the results are compared and H. B. W. is obtained by following equation.

$$H. B. W. = \min. \text{ of } (H. B. W._{\cdot a},\ H. B. W._{\cdot b},\ H. B. W._{\cdot c}) \tag{5-36}$$

This step is the final step for the investigation of H. B. W.. If the analyst wants to know the numerical ordering of nodes of the graph, he goes to following one more step.

[Step-9]. Numerical ordering of nodes in graph.

Nodes of the reordered graph, which has the minimum value of H. B. W. and selected at step 8, is labeled in accordance with the direction which is explained in previous section.

As far as the graph is filed in the prescribed filing field, the maximum difference of two nodal numbers coincides with the maximum number of nodal rows.

5-3-7. Approach to Bandwidth Reduction Method for Tree Graph with Multi-Centres

From section 5-3-3 to 5-3-6, the author proposed and explained a bandwidth reduction method, so called Sequential File Method, which can acurately lead us to the minimum value.

The characteristics of the method are i). the bandwidth of a matrix being equivalated to the width of a graph by the introduction of the filing field, ii). the optimum state of numerical ordering of nodes being recognized by the stable state of filed nodal sequences, i.e. the narrowest width of transformed graph in the field, and iii). the clearness by graphical expression of the result.

By the exact application of the method, the analyst can hope to obtain the minimum value of bandwidth for the graph being treated.

As far as a graph contains less than three centres, we may obtain the exact technique of filing only for one case, but if it contains more than four centres, we should consider on more than two cases. For example, if there is a tree graph with five centres, there arise three cases and for every case, the technique to reduce the bandwidth has to be established. (See Fig. 5-23) Furthermore, as the graph contains more centres, it includes more nodes. Thus, the problem becomes more and more complicate. This is caused by the treatment of the nodal sequence one by one.

This treatment becomes merit for rather simple problem as shown in previous sections and for them it can lead to true optimum. But for a complex problem the method becomes to be troublesome. Thus, simpler method is hoped to be found instead of treating every nodal sequence.

Following this, the author proposes the method how to overcome the complex problem.

1). Application of Sequential File Method only to a part of graph.

In a glance or only by the inspection of the original tree graph, the analyst can often find out the widest part of the graph around where a lot of nodal sequences are gathered. By the introduction of filing field, it becomes obvious that the width of a graph concerns directly to the bandwidth of the stiffness matrix of the graph. In order to reduce the bandwidth we treat only the part of the graph where the width seems to be maximum. We cut off around the part and apply the sequential file method to the part. The result may suggest near value of the true minimum one.

2). Simplification of Sequential File Method.

As described in this chapter, the complexity of the method is caused by the filing procedure of nodal sequence one by one. To remove the complexity, a nümber of nodal sequences should be treated at a time. For tree graph, centres are the most important and most effective factor for the investigation of half bandwidth. Thus, all of the nodal sequences which are connected to a centre are gathered in a rectangular area whose height and laternal length are changable but the area is constant. If the total of the nodal sequences connected to the i-th centre is equal to S_i, the area of the rectangle is also equal to the value. And the height of the area is, at least, larger than the half of the degree of the i-th centre, and less than the total of the nodes. Thus,

$$m_i - \left[\frac{m_i}{2}\right] \leqslant h_i \leqslant S_i \tag{5-37}$$

The average lateral length of the area, denoted by b_i, is calculated as following.

$$\frac{S_i}{m_i - \left[m_i/2\right]} \leqslant b_i \leqslant 1 \tag{5-38}$$

This calculation is done for every centre in the graph. After the calculation we file the area in the filing field without occupying a part of the filing field by two rectangles from neighbouring centres. At the same time, we try to minimize the height. The height presents, of course, the bandwidth of the graph.

In the procedure, the sequential file method is partially applied to the area where a part of two rectangules are mapped. That is, among the two rectangles a number of longest nodal sequences are removed from there, till the occurence of double images in the field disappears. And the removed nodal sequences are additionally filed above the area occupied already in the field.

These two methods must be furthermore investigated. If a graph is complicated and it is difficult to find out the critical part of the graph, i.e. the widest part, the first method cann't be applied without the additional operation to find out the part. In the sense, the second method becomes effective.

By the introduction of the second method the outline of the graph is easily imaged and the critical parts are picked up. After the selection of the parts, the first method is applied and we obtain the result.

As far as an analyst recognizes the outline of an objective graph, the numerical ordering of nodes in accordance with the information does not lead him to the worst result but to near optimum result. In the sense, the importance of the above method may be appreciated but more investigations and studies are required.

5–4. Application of Sequential File Method to Tree Graph

In this section the author presents some examples of the application of Sequential File Method to tree graphs in accordance with the specification in previous sections. These examples make the procedure of the method clear.

Examples for tree graph with only one centre is already shown in the previous explanation of the method. Here, the author shows some examples of tree graphs with more than two centres.

[Example–1]. Example of Tree Graph with Two Centres.

The first example is as shown in Fig. 5–24–a. Two centres are denoted by the numbers, i.e. 2 and 3, and the distance between them is that $d(2, 3) = 3$. Including the line between them, their degrees are 15 and 13, respectively. Their lengthes of the nodal sequences are presented in the figure.

The longest sequences from 2 and 3 are, at first, selected among m_2 and m_3, and they are expressed by two lines from 2 to 1 and from 3 to 4, respectively. Thus, the diameter of the graph, denoted by d_0, is equal to 17, and it is shown in Fig. 5–24–a by a thick line. These operations are the preparatory work for Sequential File Method.

Among the nodal sequences from two centres, the longest ones are selected one by one and filed in the field up to h_2 and h_3 in eq. (5–9) For this example,

$$h_2 = 7, \qquad h_3 = 6$$

These values are called the initial width. At this stage, the field may be filed up to the 7-th nodal row, and on the 3-centre one more nodal sequence is filed additionally.

The nodal capacity, Cap. $2 \sim 3$, at this initial state is calculated as following.

$$\text{Cap.}\, 2 \sim 3 = 21 \quad \text{nodes.}$$

And the residual sequences contain 40 nodes. By comparing Cap. $2 \sim 3$ and the residual nodes, it is obvious that the nodal capacity is not enough to file all of the residual nodal sequences. Thus, two more longest sequences must be filed above the left side of 2-node and also on the right side of 3-node, in order to reduce the residual sequences and also to increase the nodal capacity. And it yields that

$$\text{Cap. } 2 \sim 3 = 21 + 6 = 27$$

$$\text{Residual nodes} = 22$$

This suggests that the residual sequences may be filed in the area between two centres, and the result is shown in Fig. 5–24–b. From this final result we know that

$$\text{H. B. W.} = 10 + 1 = 11$$

The numerical ordering is given in accordance with the definition of the filing field, that is, along the direction of arrow shown in Fig. 5–2.

The final state of the filing includes unoccupied area between two centres and the nodal rows seem to be reduced. If we want to reduce one nodal row, 8 nodal spaces must be reserved at the central part. Therefore, the further reduction of rows, i.e. of H. B. W., is impossible and the figure in (b) shows the true final state and it gives one of the optimum numerical ordering of nodes for this case.

As far as the number of rows don't be increased, alternative interchanges of nodal sequences are allowed and it suggests that there exist a lot of optimum states for nodal labeling.

[Example 2]. Example of Tree Graph with Three Centres.

Fig. 5–25 shows a tree graph with three centres and 144 nodes. The centres are denoted by the numbers, i.e. 2, 3 and 4, and their degrees are 11, 9 and 10, respectively. They are denoted by $m_{i,(i=2,3,4)}$. By the selection of the longest nodal sequences from 2- and 4-centre, the diameter is constructed as presented by a thick line in Fig. 5–25. m_i presents the degree of the i-th centre except two nodal sequences which are already selected for the diameter.

The initial height for every centre is calculated and is denoted by h_2, h_3 and h_4. $d_0 = 22$ presents the length of the diameter. Fig. 5–25–a is a comparison table and is formed in order to compare the residual sequences and to select a pair of two nodal sequences which are to be filed at every additional nodal row. For this example, the table is used twice and two pairs of sequences are selected and are marked by * and ** . * and ** mean the first and the second selection, respectively. The marks are given in Fig. 5–25–b, too. Every mark coincides with the nodal sequences in Comparison Table.

The final state after the filing is expressed in the same figure, and the procedure till the state is tabled in Calculation Table(c). Step–4 and Step–5 in the table coincide with the explanation in previous section. $\bar{S}_i$ gives the node numbers included in the residual sequences from the i-th centre. S_{total} is obtained by summing $\bar{S}_i$. The column of Cap. gives the nodal capacity between 2- and 4-centres. ΔS is the difference between S_{total} and Cap., and as soon as ΔS gives positive number, the area reserves enough nodal capacity for the residual sequences. It suggests that all of the sequences may be filed in the area. And the state of the last row in the table(c) is figured in (b). From Fig. 5-25-c, we know that the half bandwidth may be resuced to 9. This value is secured by Fig. 5-25-b and -c.

[Example-3]. Example of Tree Graph with Three Centres.

Fig. 5-26 is also a tree graph with three centres. This graph contains 402 nodes. The degrees of three centres are all the same and they give the same initial heights for three centres, i.e.

$$h_2 = h_3 = h_4 = 7.$$

At step-5 in Calculation Table, eight times repetitions are needed to obtain the optimum state for nodal labeling. Every pair of selected sequences are labeled by numerical number from 1 to 8. All of the procedures to the optimum filing state are just same to Example 2. The final state is given in Fig. 5-26-b and we know the half bandwidth being reduced to 17 by counting the number of nodal rows which are needed in the filing field. Thus,

$$\text{H. B. W.} = 16 + 1 = 17.$$

[Example-4]. Example of Tree Graph with Four Centres in Series.

Fig. 5-27 shows an example of a tree graph with four centres, i.e. C_1, C_2, C_3 and C_4. The outline of the graph is given in (a). We select C_1, C_3 and C_4 as the nodes constructing the diameter for this tree. Among C_1 and C_3 nodal sequences, longest two sequences are selected, and they construct the diameter, d_0, with d_1 and d_2.

$$d_0 = 10 + d_1 + d_2 + 8 = 30$$

In accordance with the optimization procedure for four centres in previous section, we have to obtain the optimum filing for three centres. For this example, we treat, at first, C_1, C_2 and C_3 centres. Fig. 5-27-b gives the final state of filing of these three centres. At this stage, all of the nodal sequences from C_4 are removed and neglected for the filing. From the result we obtain that

$$\overline{\text{H. B. W.}} = 7 + 1 = 8.$$

Above this state of filing, we continue to file C_4-nodal sequences additionaly and the simple filing of the sequences from C_4 are done as shown in Fig. 5-27-c. That is, the additional sequences are, at first, filed in the concave area of the result for three centres, and the residuals are filed over $\overline{\text{H. B. W.}}$. The final result suggests that

$$\text{H. B. W.} = 8 + 1 = 9$$

This teaches us that this value is the minimum one and we need not reorder any sequence from C_1, C_2 and C_3. That is, the value, 8, is the minimum value for the tree graph with three centres, and the concave area has only 20 unoccupied places for C_4-sequences, and C_4 contains 35 nodes. Thus, they need additional nodal row above $\overline{\text{H. B. W.}}$. Fig. 5–27–c contains one more nodal row than Fig. 5–27–b has. We conclude that the value, 9, is the true minimum and the filing state in (c) is one of the optimum filing states for this example.

This example is a special case of a tree with four centres in which the result of three centres is just kept for the investigation of the optimum state with four centres.

[Example–5]. Example of Tree Graph with Four Centres in Series.

Second example of a tree with four centres shows general procedures for the case. Fig. 5–28 presents the case and the graph includes 338 nodes including four centres, namely C_1, C_2, C_3 and C_4.

C_1, C_2 and C_3 are, at first, selected and they construct the diameter, d_0, with the longest nodal sequences from the first and the third centres.

$$d_0 = 10 + d_1 + d_2 + 10 = 32.$$

The degrees and nodal sequences from all centres are given in Fig. 5–28–a. At first, the graph is treated as the one with three centres except C_4-nodal sequences, and the final state of filing is shown in Fig. 5–28–b.

$$\overline{\text{H. B. W.}} = 9 + 1 = 10$$

Simple additional filing of C_4-sequences above $\overline{\text{H. B. W.}}$ is followingly done and the result is presented in Fig. 5–28–c. Counting the number of nodal rows yields that

$$\text{H. B. W.} = 13 + 1 = 14.$$

But, if we introduce the steps in previous section for a tree graph with four centres, some nodal sequences from C_1 and C_3 must be reordered in the new field and the final state with the optimum ordering for C_1, C_2, C_3 and C_4-nodal sequences is as shown in Fig. 5–28–d. The number of nodal rows is reduced by one than the simple filing above $\overline{\text{H. B. W.}}$ with three centres and we know that

$$\text{H. B. W.} = 12 + 1 = 13.$$

In order to obtain this result, a nodal sequence from C_1 is replaced from the left side area of C_1 to the right side and two from C_2 are also newly refiled into the right side area of C_2. These operations are done in order to reserve wider unoccupied spaces for C_4-nodal sequences and also to leave a number of shortest sequences from C_4 to be filed in the concave area.

[Example–6].

This example is presented in order to compare the result by Sequential File Method with the other algorithms which were proposed by E. Cuthill & J. McKee,[39] I. P. King[45] and

Levy. The graph is a tree one, and the results obtained by the other algorithms are devived from Cathill's paper.[40] It is obvious that the sequential file method can give the minimum bandwidth among the results, and the value is the true minimum for the system. "Profile" means the summation of the area of stiffness matrix between the main diagonal and the first non-zero element for every row matrix. Profile obtained by the author's method, is rather better comparing with the others, though the sequential file method does not aim to minimize it. About the profile minimization method the author gives some considerations in Chapter 8 of this thesis.

5–5. Conclusions.

In these investigations, the new method for bandwidth reduction is, at first, introduced. The method is available only for tree structures, which have distinguished configurations comparing to general civil engineering structures.

The use of tree graph may clarify the unkown factors which are influent on the bandwidth.

By the separation of filing field and the filing techniques, the bandwidth reduction is transformed into how to reduce the width of the graph in the filing field. Therefore, this chapter contributes to show the new method to draw the original graph with the narrowest width in the field.

The method is explained in accordance with the number of centres where more than three lines are connected.

In example 6 in the last section the author compared the sequential file method with the other algorithms. These algorithms cann't give good results for any tree system except the case with one centre and deg. = 3. That is, after finding the initial node the node being labeled "2" is selected among those nodes which are located from the initial node by d = 1. This procedure is general for any algorithm except the sequential file method. But, it is obvious that the procedure cann't lead to true minimum bandwidth.

As far as a graph contains only a few centres, the minimum bandwidth is easily obtained even if the system has more than hundreds or thousands nodes. But, as the number of centres increases, the application of the proposed method may become troublesome, because every nodal sequence in the graph is treated one by one in order to place it in the filing field and the procedure necessarily includes as many filing operations as the number of nodal sequences. Furthermore, the process of filing nodal sequences in the field requires the comparison of all nodal sequences which are not yet filed at the stage.

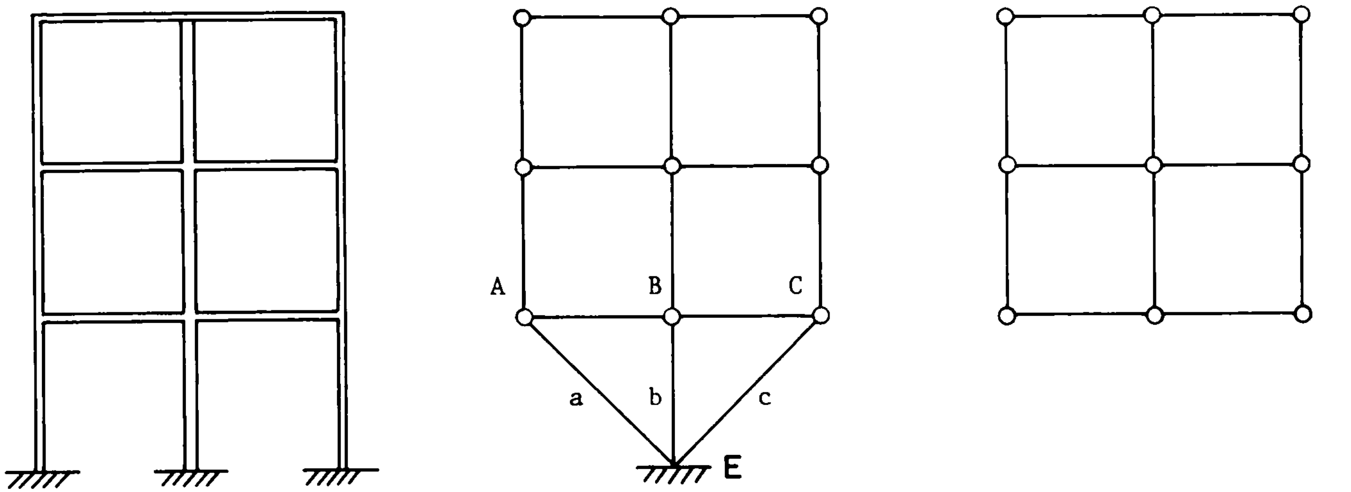
Therefore, simpler methods are needed and one of them is also proposed in this chapter. In the method, the analyst does not treat nodal sequences but he treats the nodal sequences as an area which includes the same number of nodes as the sequences contain. But, only the outline of the method is shown here and the details must be studied in future.

The explanations of the sequential file method seem to be very complicated for the application to acutal systems. But, if it is applied acutally, some steps in the method become unnecessary and they may be removed. The complexity of the sequential file method comes from that the method can be applied to any tree graph and it can induce the true minimum value of bandwidth of the graph.

Furthermore, the method is a graphical one and after the filing of all the sequences in a graph, the state of the filed configuration certificates whether the result gives the minimum value of bandwidth. At the same time, the distinguished property follows to the demerit of the method. That is, as far as a system is treated as a graph, the number of nodes which can be treated at a time may be restricted. In the sense, the method proposed in the last section will become important.

In past studies tree systems are scarcely treated, because the structure corresponding to tree system need not be given the optimal nummerical ordering by the reason that it is a kind of statically determinate system. But, the kind of structure has a distinguished topological property and by the aid of the inspection for the kind of structure, the factors which give influence to the bandwidth reduction may be clarified.

Moreover, a kind of general civil engineering structure with meshes corresponds to tree system and the results which were obtained in this chapter become available and useful for the kind of structures. The author gives the investigations with respect to this fact in following chapter.



(a) Structural Framed System (b) Linear Graph (c) Graph for Nodal-Labeling

Fig. 5–1 A Framed System and its Linear Graph

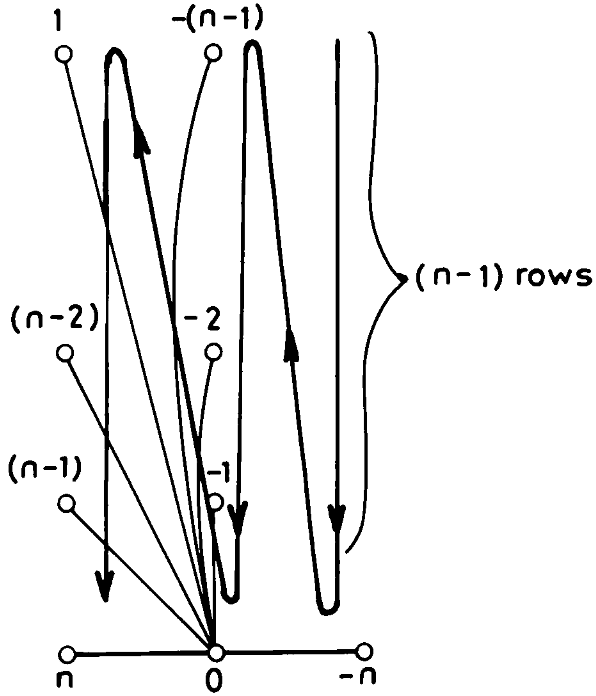
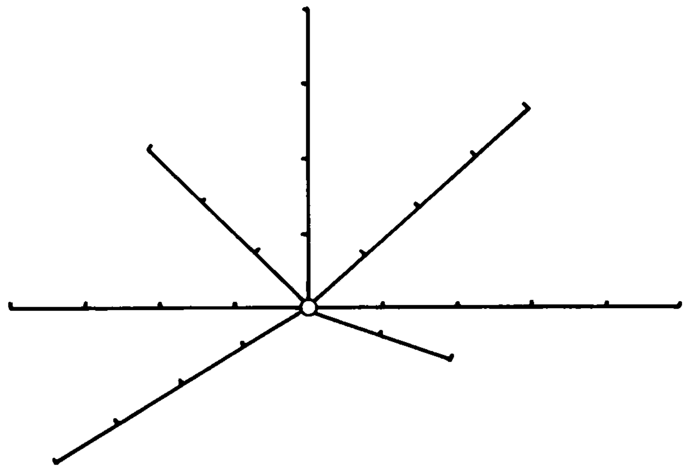
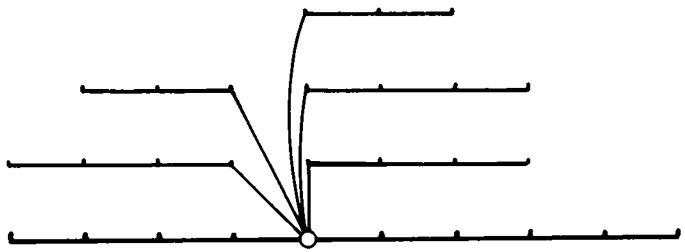


Fig. 5–2 Arrangement of Nodes around a Node with Multi Degree



(a) A Tree Graph (A Node with Multi Degree)



(b) Graphical Expression

Fig. 5–3 Graphical Determination of Half Bandwidth

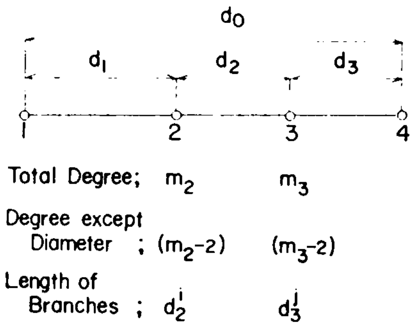


Fig. 5–4 An Example of Tree Graph with Two Centres

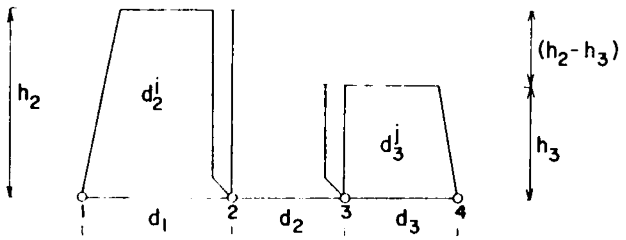


Fig. 5–5 Initial State of Sequential File Method for Tree Graph with Two Centres

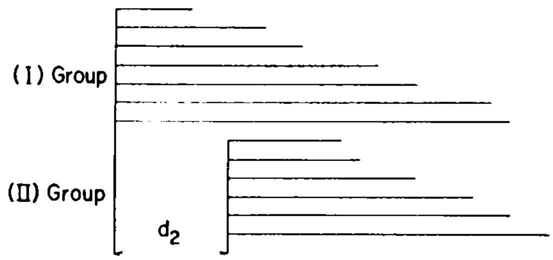


Fig. 5–6 Comparison Table for Residual Sequences

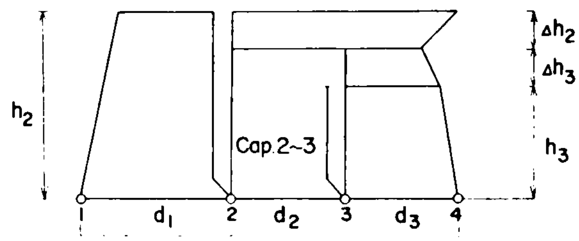


Fig. 5–7 Last State of Sequential File Method for Tree Graph with Two Centres

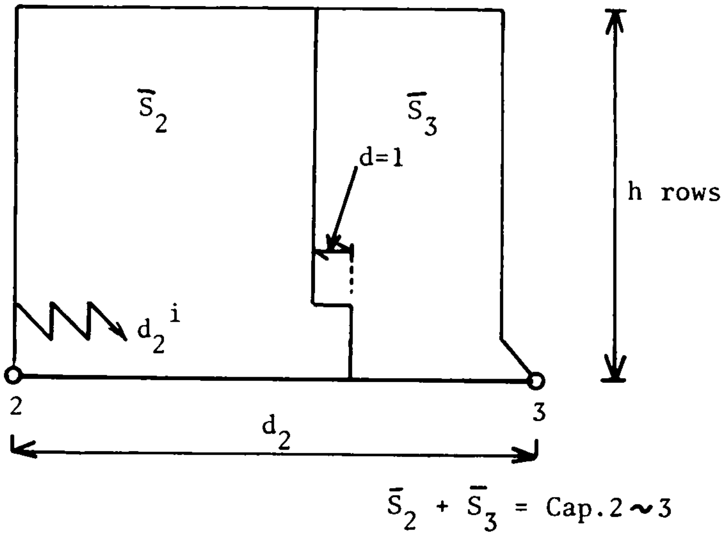


Fig. 5–8 How to File Cap. 2 ~ 3

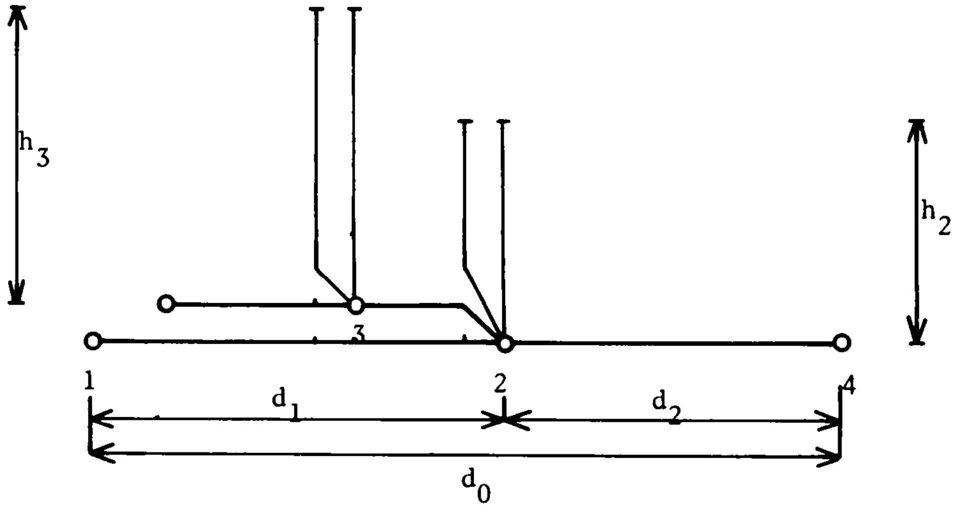


Fig. 5–9 An Example of Sequential File Method with True Diameter ($h_3 > h_2$)

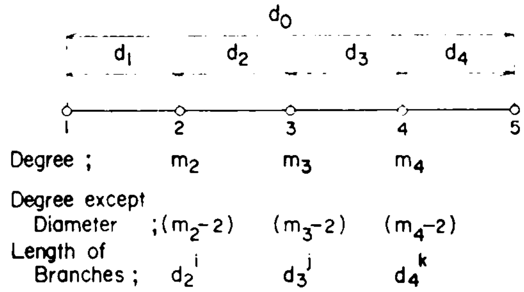


Fig. 5–10 An Example of Tree System with Three Centres

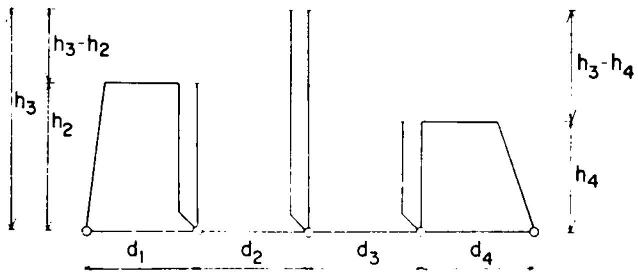


Fig. 5–11 Initial State of Sequential File Method for Tree System with Three Centres

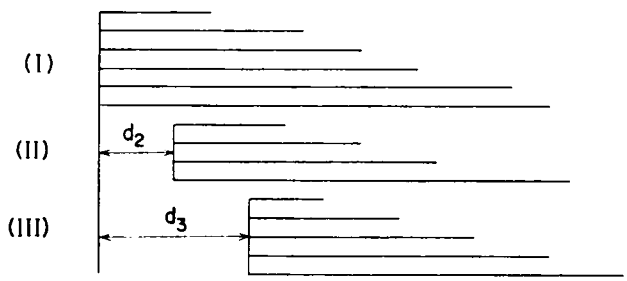


Fig. 5–12 Comparison of Residual Nodal Sequences

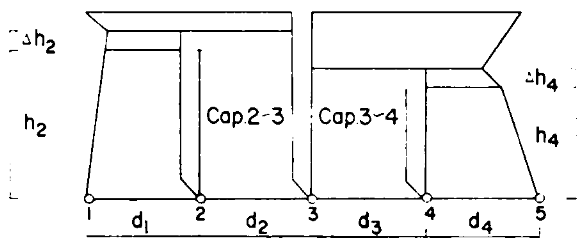


Fig. 5–13 Sequential File Method for Tree System with Three Centres at the Final State

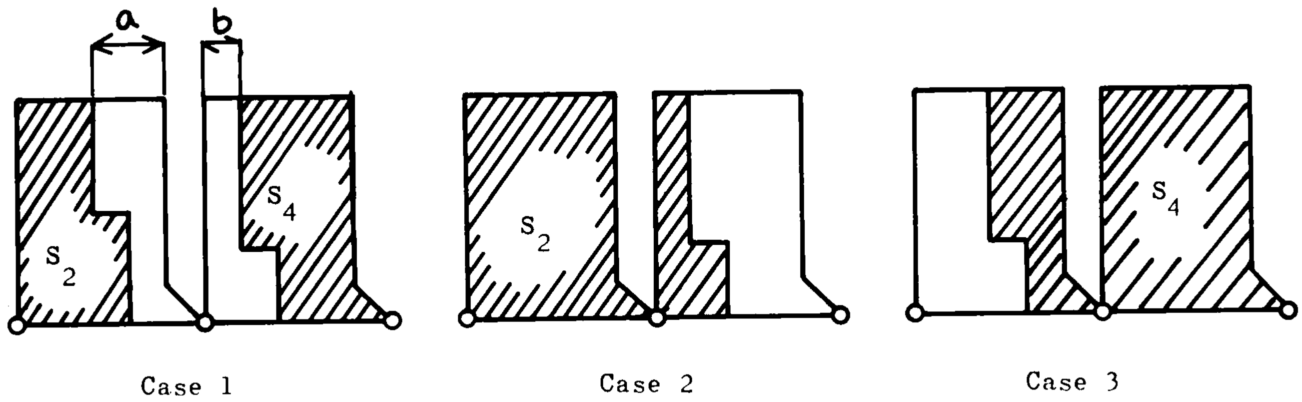


Fig. 5–14 Possible Cases at Step–5 of Sequential File Method

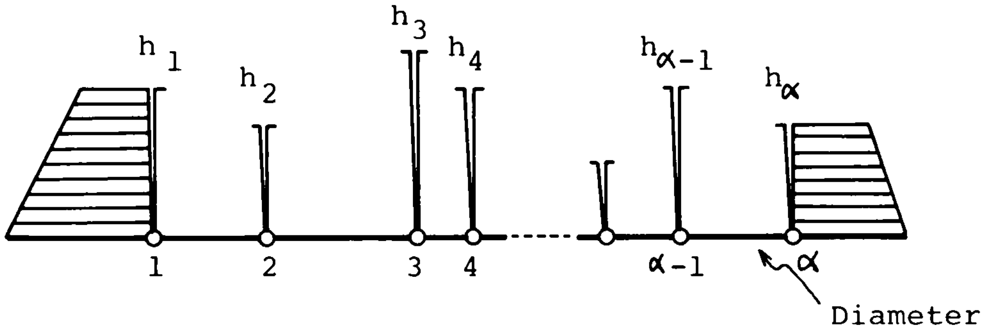


Fig. 5-15 Initial State of Sequential File Method for Tree Graph with Multi Centres in Series

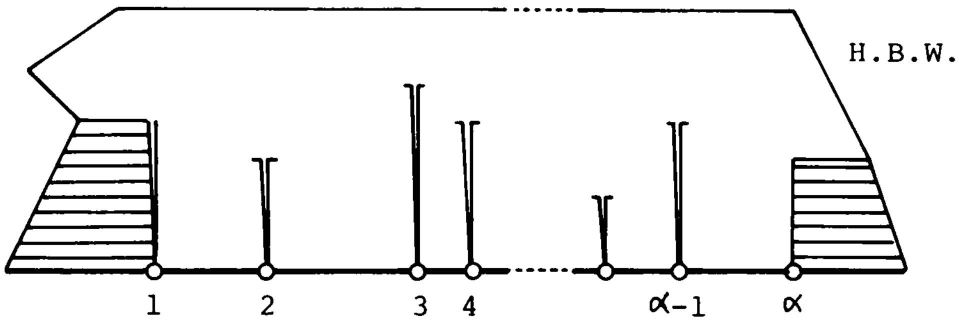


Fig. 5-16 Final State of Sequential File Method for Tree Graph with Multi Centres in Series

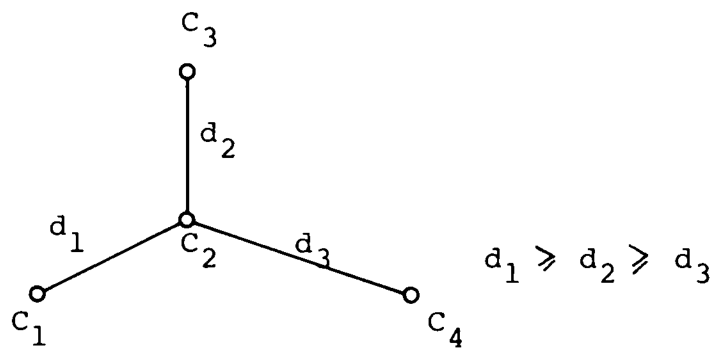


Fig. 5–17 A Tree Graph with Four Centres

Centre	Degree	Length of Nodal Sequences
C_1	m_1	$l_1^1,\ l_2^1,\ l_3^1,\ l_4^1,\ \ldots,\ l_{m_1}^1$
C_2	m_2	$l_1^2,\ l_2^2,\ l_3^2,\ l_4^2,\ \ldots,\ l_{m_2}^2$
C_3	m_3	$l_1^3,\ l_2^3,\ l_3^3,\ l_4^3,\ \ldots,\ l_{m_3}^3$
C_4	m_4	$l_1^4,\ l_2^4,\ l_3^4,\ l_4^4,\ \ldots,\ l_{m_4}^4$

$$l_j^i \geqslant l_{j+1}^i \qquad \left|\begin{array}{l} i = 1, 2, 3, 4 \\ j = 1, 2, \ldots, m_i \end{array}\right.$$

Table 5–1 Characteristics of A Tree with Four Centres

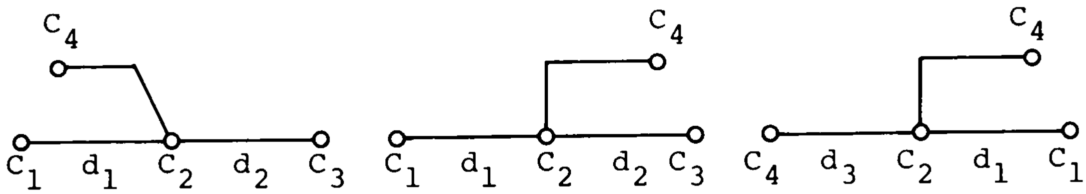


Fig. 5-18 Possible Temporary Diameter for A Tree with Four Centres

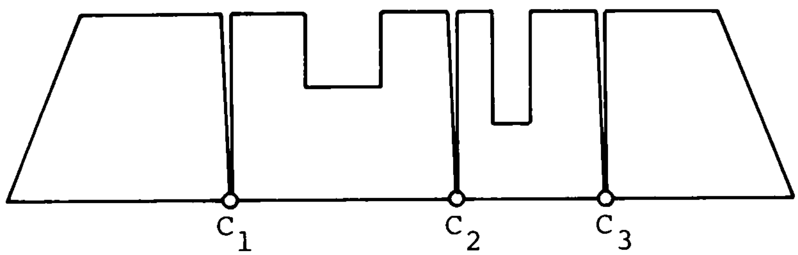


Fig. 5-19 Final State of Sequential Filing for Tree Graph with Three Centres

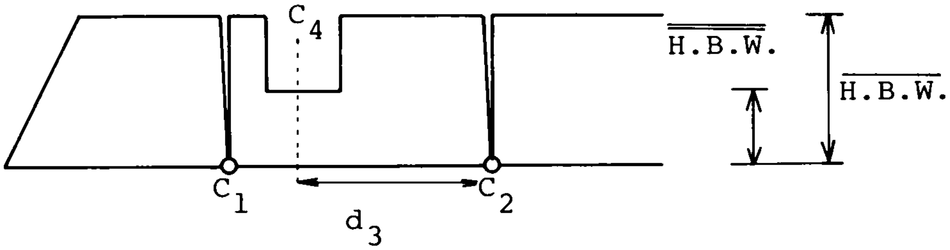


Fig. 5-20 Relation between Concave Area and The Distance to The Fourth Centre

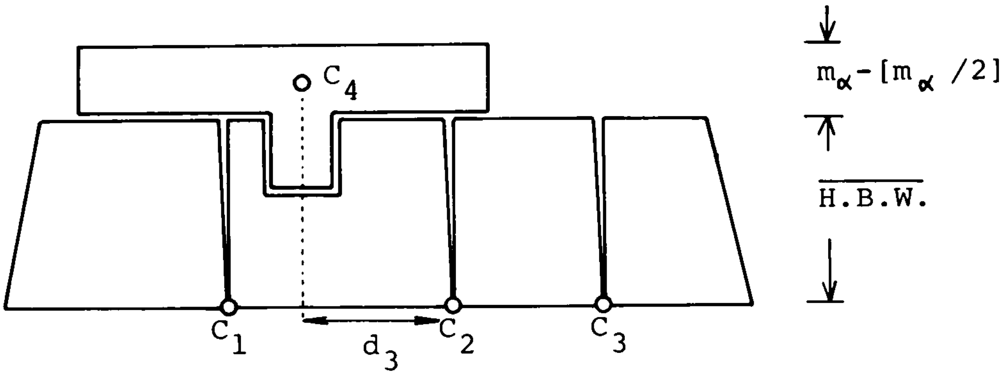


Fig. 5-21 Simple Filing of Sequences from The Fourth Centre above The Result for Tree with Three Centres

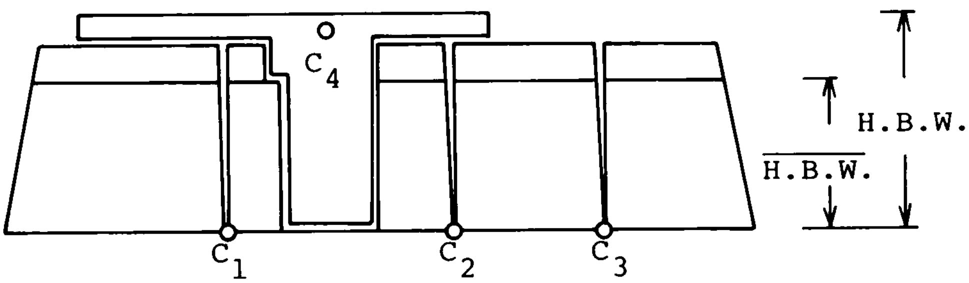


Fig. 5-22 Final Filing State of Tree Graph with Four Centres

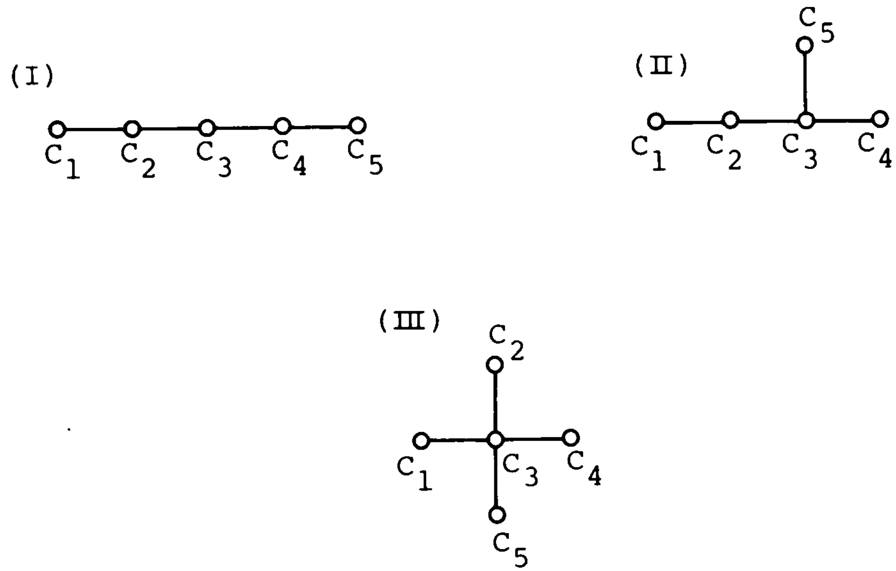
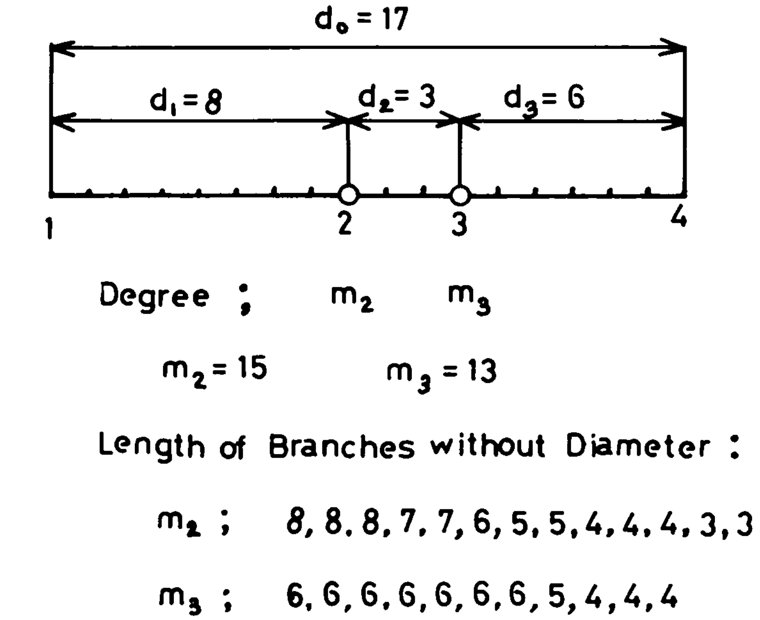
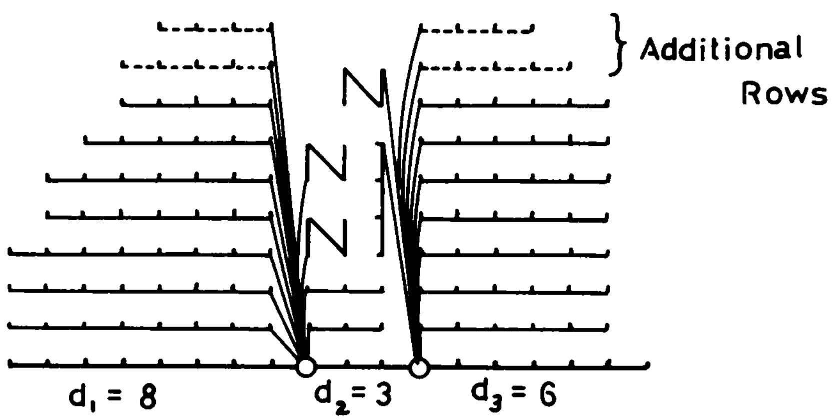


Fig. 5-23 Three Cases for Tree Graph with Five Centres

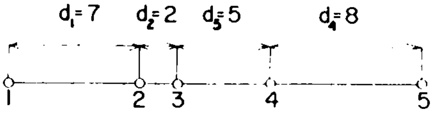


(a) A Tree of Two Nodes with Multi Degree



(b) Graphical Expression

Fig. 5-24 Graphical Determination of Half Bandwidth



$\bar{m}_i = m_i - 2$

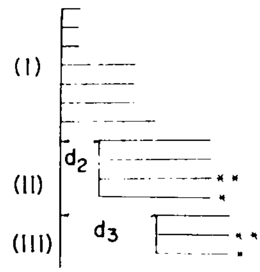
$\bar{m}_2 = 9$; $d_2^i = (7, 7, 7, 7, 6, 6, 6, 6, 6)$

$\bar{m}_3 = 7$; $d_3^j = (5, 4, 4, 4, 1, 1, 1)$

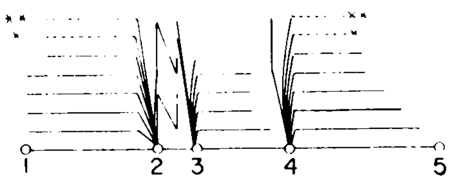
$\bar{m}_4 = 8$; $d_4^k = (8, 7, 6, 5, 5, 4, 4, 4)$

$h_2 = 5$, $h_3 = 4$, $h_4 = 4$

$d_0 = 22$



(a) Comparison of Sequences

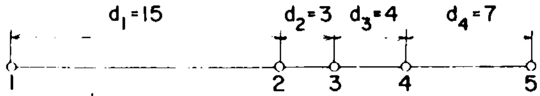


(b) Sequential File Method

		$\bar{S}_2$	$\bar{S}_3$	$\bar{S}_4$	S total	Cap.	ΔS
Step-4		24	20	12	56	35	− 21
Step-5	*	6	0	4	10	+ 7	− 4
		18	20	8	46	42	
	**	6	0	− 4	− 10	+ 7	+ 13
		12	20	4	36	49	

(c) Calculation Table

Fig. 5−25 An Example of Sequential File Method



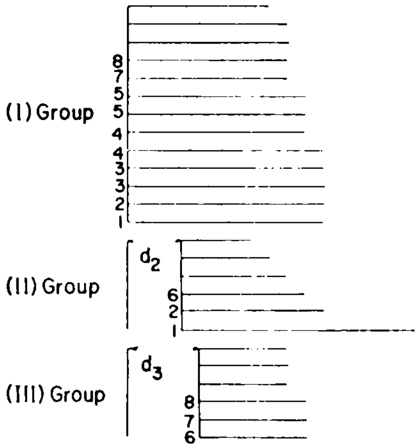
$\bar{m}_2 = 13$; $d_2^i = (15, 15, 15, 14, 14, 14, 13, 13, 8, 7, 6, 5, 4)$

$\bar{m}_3 = 13$; $d_3^j = (11, 11, 11, 11, 11, 10, 10, 10, 9, 9, 9, 9, 8)$

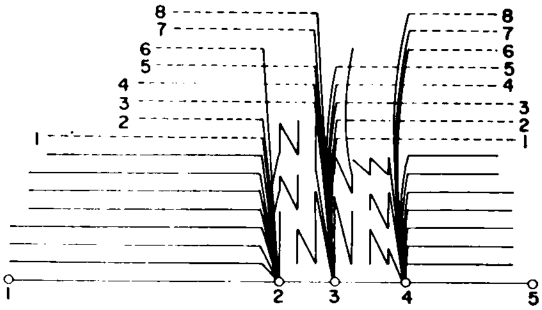
$\bar{m}_4 = 13$; $d_4^k = (7, 7, 7, 7, 7, 6, 6, 6, 6, 6, 5, 5, 5)$

$h_2 = h_3 = h_4 = 7$

$d_0 = 29$



(a) Comparison Table of Residual Nodal Sequences

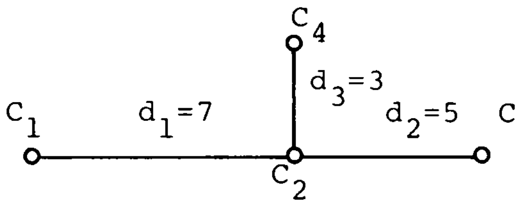


(b) Sequential File Method

		$\bar{S}_2$	$\bar{S}_3$	$\bar{S}_4$	S total	C ap.	ΔS
Step-4		43	129	33	205	49	−156
Step-5	1	30	118	33	181	52	−129
	2	22	107	33	162	55	−107
	3	22	85	33	140	55	−85
	4	22	64	33	119	55	−64
	5	22	44	33	99	55	−44
	6	15	44	27	86	62	−24
	7	15	35	21	71	66	−5
	8	15	26	15	56	70	+14

(c) Calculation Table

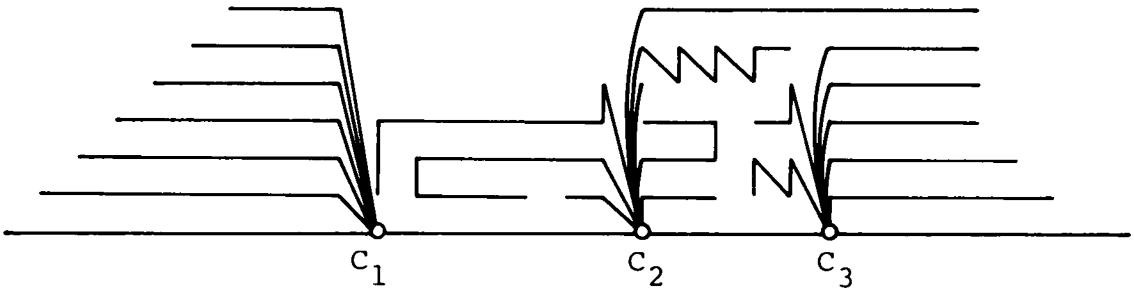
Fig. 5−26 An Example of Sequential File Method



m_1= 8; 10, 9, 8, 7, 6, 5, 4, and 7
m_2=11; 10, 10, 10, 8, 6, 3, 2, 1, and 7, 5, 3
m_3= 9; 8, 7, 6, 5, 5, 5, 4, 3, and 5
m_4= 8; 10, 8, 6, 4, 2, 1, 1, and 3

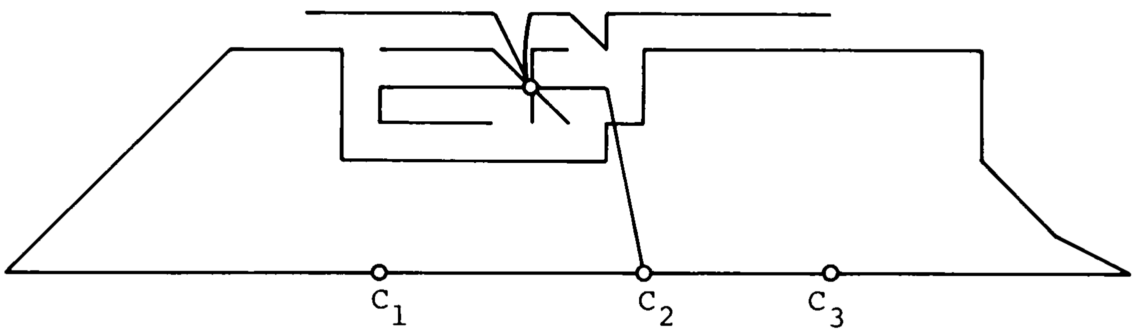
Total Number of Nodes = 190

(a) Given Graph



H.B.W.=8

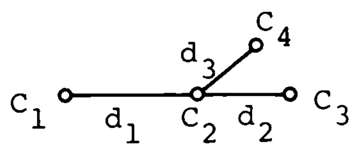
(b) Optimum Filing State for Tree Graph with Three Centres



H.B.W.=8+1=9

(c) Simple Filing of C_4-Sequences above (b)–Result

Fig. 5–27 An Example of A Tree Graph with Four Centres



$d_1=7$, $d_2=5$, $d_3=4$

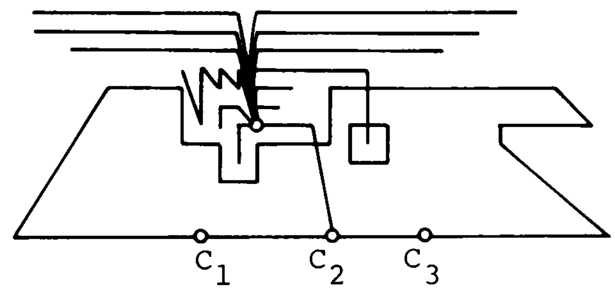
$m_1=14$; 10, 10, 10, 10, 8, 8, 7, 6, 5, 4, 3, 2, 1, 7

$m_2=10$; 15, 14, 13, 12, 11, 8, 6, 7, 5, 4

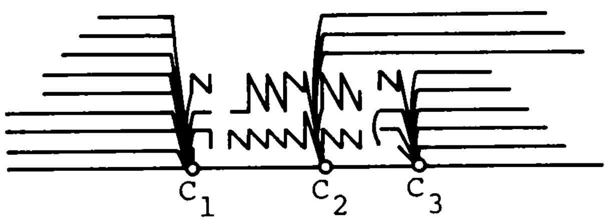
$m_3=11$; 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 5

$m_4=13$; 15, 14, 13, 12, 11, 10, 9, 8, 3, 3, 3, 2, 4

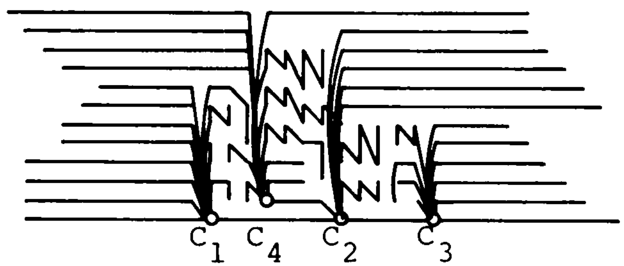
(a) Given Tree Graph



(c) Simple Filing above (b)–Result

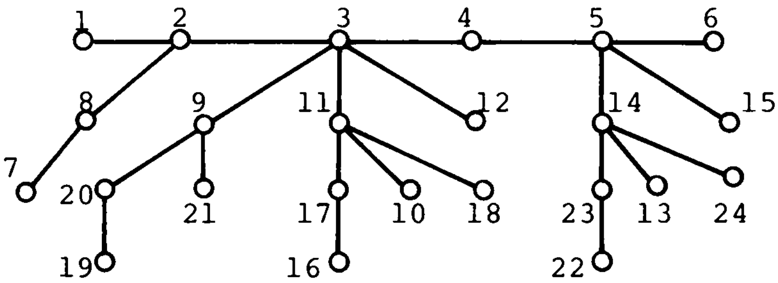


(b) Tree Graph with Three Centres

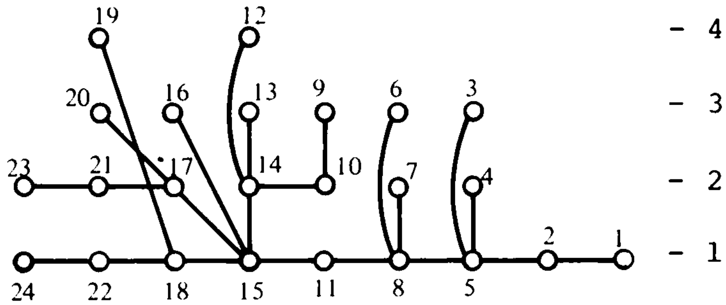


(d) Optimum Result for Four Centres

Fig. 5–28 An Example of A Tree Graph with Four Centres



(a) Original Labeled Tree Structure



(b) Result of Sequential File Method (H. B. W. = 5)

Algorithm	Bandwidth	Profile
Original	13	107
Cuthill-McKee	10	101
King	13	59
Levy	14	37
Sequential File Method	5	41

(c) Comparison of Results by Several Algorithms

Fig. 5-29 An Example of Tree Structure with 6 Centres

CHAPTER 6
APPLICATION OF SEQUENTIAL FILE METHOD TO MESH STRUCTURES

6–1. Introduction

Sequential File Method is developed only to be applied to tree structures whose bandwidth is wanted to be found. In this chapter, the author trys to apply the method to mesh structures.

Mesh graphs have quite different topology comparing to tree graphs, that is, there exist more than two pathes between two nodes in the former graphs, while only one path exists for the latter.

Filing the graph in the two-dimensional filing field, all of the distances of pathes from a node to the other nodes have to be held in the field. Therefore, as the number of pathes from a node increases, the location of the node in the field is restricted and affected by the locations of the other nodes. In the sense, the degree of freedom of a node in mesh is less than that of tree graph. Thus, the direct application of the sequential file method to mesh graphs is obviously difficult.

The difficulty does not occure from the use of the filing field, because it allows any nodal ordering. It is caused by the filing technique of graphs in the field.

For tree graph there exists only one path between every two nodes and the distance between them is an important factor to decide the location of these two nodes in the filing field. That is, along the lateral axis in the field they must be located within the value.

In general type of mesh graphs, there exist a lot of pathes between two nodes. But only the shortest pathes among them may restrict the maximum distance between them. In the sense, a kind of mesh graphs contain rather few pathes which correspond to the shortest path, even if the system contains a lot of pathes between every two nodes. The characteristic of these graphs seems to be similar to the characteristic of tree system and the method may be applicable.

If a part of a mesh system is connected to the main part of the system only by few lines, the number of the shortest pathes between two nodes in different parts are restricted by the number of the lines connecting the two parts, and only a part of the whole graph can effect the distance between them. In other words, the degree of freedom of the nodes in the filing field is rather large enough as nodes in a tree graph have, though the location of a node included in usual mesh in the filing field is almost defined by the setting of the other part of the graph in the field.

If a mesh graph can be divided into trunk part, where the diameter of the graph exists, and branch parts, they may be transformed into the diameter and the other branches of a tree graph by modifying the original mesh graph.

In this chapter, the author, at first, trys to apply the sequential file method to some types of mesh structures without modifying the method but transforming the structures, themselves. There, the author does not introduce any new factor in the method proposed in previous chapter, and he investigates the possibility of the method applied to meshes.

The examples given in the part correspond to tree graphs with some centres in previous chapter, and the explanations of the application of the method are done for each example.

In order to treat the actual mesh systems, the sequential file method must be modified. But as far as we treat mesh systems whose diameters are obvious, we may not fail to give bad numerical ordering of nodes but can give rather good results. The filing field becomes powerful tool for the judgement of the results.

The latter half of this chapter contributes to give a reduction method of half bandwidth of mesh systems which correspond to actual structures with evident diameters. Thus, the examples given in the section are bridge structures whose longitudinal axes present the diameters.

6–2. Direct Application of Sequential File Method to Mesh Structures.

In this section, the sequential file method is applied to a type of mesh structures whose boundary configurations are rather complicated but which can be transformed into tree graphs.

[Example–1]. Simple mesh graph corresponding to tree graph with two centres.

Fig. 6–1–a is an example of framed structure. If we present a mesh by a node, the system can be reduced to a tree system in Fig. 6–1–b. We can apply the sequential file method for the tree system and the result is shown in Fig. 6–1–c. Even if we give the numerical numbers to the filed tree graph, the number is actually attached to mesh. According to the order of meshes indicated in Fig. 6–1–c, we should give proper numerical number for joints in every mesh, successively.

In this example, the mesh graph is modified into a tree graph with two centres. (c) shows the last state for the tree and we know only the outline of the optimum state of the original graph from the result. As far as we treat the modified graph, the sequential file method is strictly applied, but at the stage of obtaining the exact result for the original graph from the tree one we have to pay attention to the part of the tree where a nodal sequence is bent.

[Example–2]. Mesh graph corresponding to tree graph with two centres.

Fig. 6–2 shows an example of mesh graph with 106 nodes. It is also modified into a tree graph with two centres as shown in Fig. 6–2–b. The result of tree graph is applied to obtain the half bandwidth of the original graph as shown in (c). Comparing (b) and (c), we know that one nodal row in (b) coincides with two rows in (c). Thus, by counting the number of nodal rows which are occupied by the nodes, we know that

$$\text{H.B.W.} = 6 + 1 = 7.$$

The parts of the graph which are bent in (c) are carefully treated and they are just filed as same as the nodal sequences of the tree are filed. Nodes are labeled in accordance with the specification for the filing field.

From the result in (c), we know that the half bandwidth cann't be reduced any more, because below the fourth nodal row the filing field has not enough nodal spaces for the additional filing of the mesh branch which is just filed in the 5th and the 6th nodal rows in (c). In this example it is observed that the degree of a centre for a tree coincides with the number of mesh branches.

[Example-3]. Mesh graph corresponding to tree graph with three centres.

Fig. 6-3-a gives an example of a mesh graph with 156 nodes. (b) is the final state of the filing the nodal sequences of the transformed tree graph. (c) shows the optimum state of numerical ordering for the original mesh graph which is obtained directly from (b) result. In the figure of (c), we notice that the mesh branch at the bottom does not construct the true diameter but the mesh branches which occupy the 3rd and the 4th nodal rows express the diameter. But the result in (c) gives also the minimum bandwidth, because the nodes which occupy the 5th and the 6th rows cann't be filed below the 4th row. Thus, we conclude that

$$\text{H. B. W.} = 6 + 1 = 7$$

[Example-4]. Mesh graph corresponding to tree graph with four centres.

A mesh graph corresponding to a tree graph with four centres in series is given in Fig. 6-4-a. The graph has 210 nodes. In this example, there is not given the filing state of the transformed tree graph but only the final result in (b). The mesh branch in the bottom does not construct the diameter but we can easily refile the filed branches in order to coincide the bottom one with the true diameter, if it is desired. It is obvious that the figure gives the optimum state which presents the minimum bandwidth of the mesh graph. The value of the bandwidth is obtained by counting the number of nodal rows in (c).

$$\text{H. B. W.} = 8 + 1 = 9.$$

[Example-5]. Mesh graph corresponding to tree graph with 6 centres.

Fig. 6-5-a presents a mesh graph with 176 nodes. The graph is treated as a tree graph with 6 centres and the final result of the optimum filing of mesh branches is presented in Fig. 6-5-b. This graph is a little complicated but it is obvious that the number of nodal rows cann't be reduced any more. Therefore,

$$\text{H. B. W.} = 8 + 1 = 9$$

[Example-6]. General mesh graph corresponding to tree graph.

The examples which were treated in this section are mesh graphs in which all of the mesh branches have the same width. Fig. 6-6-a is a new example with mesh branches with different width. The graph contains 101 nodes. The procedure which is applied to the previous examples is available for this case, too. The final state of filing mesh branches is shown in Fig. 6–6- b. In this case, mesh branch filed in the bottom of (c) is wider than the other mesh branches which are optimaly filed above. From the result we know that

$$\text{H. B. W.} = 7 + 1 = 8.$$

In this example the original mesh graph is treated as a tree graph with 6 centres.

From these examples it becomes obvious that some types of mesh graphs may be treated as much simpler tree graphs and the optimum numerical ordering of nodes with minimum bandwidth can be obtained by the direct use of the results of tree graphs.

The examples treated here are mesh systems consisting of only rectangular meshes. Mesh systems with other mesh forms may be treated as described in this section.

The characteristic which is common for all the examples is that the nodal distribution in the given graphs is constant for unit area of the graphs. That is, the distance between every neighbouring nodes has the same length. The characteristic makes the application of the sequential file method to graphs easy, because the analyst can find out the diameter of the graph without effort. If he can find out it, he obtains the most important factor for the reduction of the bandwidth.

As described in previous chapters, the diameter expresses the vague flow of numerical ordeing of nodes and also the location of the initial and the final nodes of the optimum ordering. The information leads the analyst to near optimum state of sequential filing. In the sense, all of the examples which are treated in this section are rather simple problems for the reduction of their bandwidthes.

If the nodal distribution in a graph is not uniform, it is very important and also difficult problem to find out which path in the graph constructs the diameter. Thus, the first treatment of the graph is to equate the lengthes of a number of lines and to make obvious where the diameter is hidden. If this treatment is successful, the analyst may not fail the reduction of its bandwidth and he obtains the minimum or near minimum value of H. B. W. of the graph.

In mesh graphs which present the actual structural systems (especialy the framed structures or truss structures), it is not so much difficult to find the pathes of the diameters as the meshes which express the topology of the structural behaviour of continuous media.

In following section, the author treats actual structural systems and shows how to reduce the value of half bandwidth by the aid of the filing field.

6–3. Treatment of Actual Framed Structures by the Aid of Filing Field

This section contributes to consider and to clarify the merits of the sequential file method being applied to actual structural systems (especially to the mesh systems) and also to show how to utilize the two-dimensional filing field efficiently.

The summarization in the last some paragraphs in section 6-2 of this chapter appoints that the examples in the section are rather simple ones for the reduction of their bandwidth, though their surrounding configurations seem to be somewhat complicated. It is caused by the constant distance between every neighbouring nodes in the graph. The fact makes us ease to find the diameter of the graph.

Observing actual graph corresponding to framed structural system, the lengthes of members are not same and it induces the difficulty of finding the diameter.

If we treat tree systems or the mesh systems similar to the tree graphs which are presented in the previous section, the selection of diameter is not so much difficult as the inspection for general meshes.

If general structures in civil engineering are treated, most of them contain a lot of meshes. Thus, it seems to be difficult to apply the method to them. But, generally speaking, actual structures have not so many complicated mesh graphs as abstract graphs can contain. Even if the system, itself, is very complicated, we may easily find out the diameter from the outline of the system.

Any structural system has a definite configuration which can satisfy the object of the construction of the system, and therefore any member in the system is displayed with a distinguished purpose. For example, if the system has an objective to get over an obstacle, the system has enough length to clear over it and other dimension, i.e. the width and the height of the structure, may be necessarily decided secondarily.

Bridge structure corresponds to the system. Obviously, the structure has dominant length along the longitudinal axis of the bridge. The length is dominant comparing to other dimensions. The axis corresponds to the diameter of the graph which shows the topology of the actual structural system. As shown in this example, the diameter of the graph is found out in a glance, even if the system may be complicated and we can have the most important factor to obtain the method of bandwidth reduction.

Furthermore, as far as we treat actual mesh structures, the degree at any joint in the system is also restricted. That is, the display of a member is decided to optimize the purpose. For example, the display of a member is restricted in order to keep space for some usage in the structure. Therefore, the number of members connected to a joint has some upper limit, necessarily.

These characteristics of actual structural systems make the numerical ordering of joints somewhat easy. But, it can be generally said that the treatment in previous section cann't be directly applied to actual structures. That is, their configurations cann't be transformed into tree graphs. Following this, the author explains the general treatment of mesh structures in order to obtain the numerical ordering of joints.

[Step-1]. Preparation of filing field.

The filing field is efficiently used for the purpose. Generally, we prepare the two-dimensional filing field.

[Step–2]. Investigation of the diameter.

The diameter of the structure must be, of course, found out.

[Step 3]. Finding the maximum width of the graph.

Here, we treat a graph whose diameter is obvious. Thus, the width of the graph (i.e. perpendicular to the direction of the diameter) can be also guessed. The maximum value and the area where the value appears teach the analyst the supposing value of the bandwidth and the location of nodal columns in the filing field where the maximum bandwidth appears. Thus, the analyst must pay attention to the area in the stage of filing the part of the graph in the filing field.

[Step–4]. Filing procedure of the graph.

The graph is filed in the field in accordance with the attentions given in previous step–3.

The graph is stretched as long as possible and the longitudinal axis of the graph is ordered along the lateral direction in the filing field. At this first filing, the direction of a line may not be restricted to the allowable directions in the field. After the filing, they must be rearranged as to obey the restrictions of directions.

[Step–5]. Refiling procedure of the image in the field.

After the step–4, the mapped graph may be long enough in vertical direction and there may exist enough unoccupied area under the graph to refile a part of the graph. If the number of the nodal rows can be reduced by the refiling, the part must be filed in the area in accordance with the allowable directions.

If the graph is refiled in γ nodal rows, the half bandwidth is calculated by following equation.

$$\text{H. B. W.} = (\text{ occupied nodal rows }) + 1 = \gamma + 1 \tag{6–1}$$

Step 5 is not neccessary but additional step for the bandwidth reduction. That is, the step is not concerned to the substance of the reduction of bandwidth. But the step may clearly express the value of the bandwidth of a graph. We can obtain the value from the result of step–4.

For every nodal column, the number of nodes is counted. We denote the number of nodes at the i-th nodal column by N_i. Then, following sequence of numbers is obtained for the graph from the result of step 4.

$$(N_1, N_2, \ldots, N_i, \ldots, N_{d_0}) \tag{6–2}$$

Following this, the difference of the locations of the top in neighbouring nodal columns is calculated. The difference is zero or plus integer number. That is, if the top of the left-side nodal column is lower than that of the righ-side one, the difference needs not be counted and it is denoted by zero. Thus, for the cases where the top of the left-side nodal column is at the same level to or higher than the right-side one, the difference is counted.

The difference between the i-th and the (i + 1)-th nodal columns is denoted by δ_{i+1}, and we obtain following sequence.

$$(0, \delta_2, \delta_3, \ldots, \delta_i, \delta_{i+1}, \delta_{i+2}, \ldots, \delta_{d_0}) \qquad (6\text{–}3)$$

, in which the first nodal column cann't be compared and the difference is equated to zero. By the addition of these two sequences we obtain

$$(N_1, N_2 + \delta_2, N_3 + \delta_3, \ldots, N_i + \delta_i, \ldots, N_{d_0} + \delta_{d_0}) \qquad (6\text{–}4)$$

We find the maximum value among the sequence and the value gives the half bandwidth.

$$\text{H. B. W.} = \{\text{Max. of } (N_i + \delta_i)\} + 1, \qquad \text{namely } \delta_1 = 0. \qquad (6\text{–}5)$$

From the above equation it is clarified that following items are important in order to reduce the bandwidth.

1. The number of nodes included in a nodal column should be reduced.
2. The difference of the tops of neighbouring nodal columns should be reduced as small as possible.

These two items suggest that the original graph should be mapped in the field as narrow as possible. But, as the graph has the limit of stretching in lateral direction (i.e. it can be, at most, stretched as long as its diameter), we obtain the lower bound of H. B. W..

$$\text{H. B. W.} \geqslant \left[\frac{n}{d_0 + 1} \right] + 1. \qquad (6\text{–}6)$$

, where n is the number of nodes in the graph and d_0 expresses the diameter.

Following above mentioned steps for the bandwidth reduction of mesh graphs, the author explains, at first, the application of the method by the aid of very simple example.

Fig. 6–7–a is an example of mesh graph and the direction of the diameter can be observed without effort. For this example,

$$d_0 = 12. \qquad (6\text{–}7)$$

Paying attention to the longitudinal direction, it is mapped in the filing field as shown in Fig. 6–7–b. In the figure, we notice that the diameter cann't be stretched in a straight line by the restriction of allowable directions. From this figure, H. B. W. is calculated by counting the number of rows and

$$\text{H. B. W.} = 3 + 1 = 4. \qquad (6\text{–}8)$$

The value is strictly the minimum half bandwidth of the graph and it is certificated by our experiences. The configuration in (b) is the one which is stretched as long as possible and the graph covers the length of its true diameter. The graphs in (c) and (d) are shorter than the configuration of (b) by one along the lateral axis of the field. But, they give the true half bandwidth and we know that they show other optimum numerical orderings of nodes. Furthermore, the original graph, itself, presents one of optimum states of mapping the graph in the field, because the number of rows is equal to three and all lines in the graph obey the restriction of allowable directions.

Observing Fig. 6 7 a, b, c and d, it is noticed that the number "2" of the numerical ordering is given to a node on the shorter edge of the graph.

Mapping the graph in the field, the configuration given in Fig, 6–7–e is also considered. Observing the figure, it seems to be stretched as long as possible. But, there are enough unoccupied area under the mapped graph to refile the mapped one. The reordered graph is presented in Fig. 6–7–f. Investigating the figure, the length of the configuration becomes shorter than that of (e), though the numerical ordering of them are just the same. Thus, it is concluded that the two configurations of (e) and (f) are equivalent each other.
From Fig. 6–7–e.

$$\begin{aligned} N_i &\equiv \{1, 2, 3, 3, \ldots, 3, \ldots, 2, 1\} \\ \delta_i &\equiv \{0, 1, 1, 1, \ldots, 1, \ldots, 0, 0\} \end{aligned} \qquad (6\text{–}9)$$

Thus,

$$\begin{aligned} \text{H.B.W.} &= \text{Max. of } (N_i + \delta_i) + 1 \\ &= 5. \end{aligned} \qquad (6\text{–}10)$$

And we obtain from Fig. 6–7–f that

$$\begin{aligned} \text{H.B.W.} &= n + 1 \\ &= 4 + 1 = 5. \end{aligned} \qquad (6\text{–}11)$$

The former result coincides with the latter. This value is larger than the results from Fig. 6–7–a, b, c, and d by one. The increase of the bandwidth is induced by the numerical ordering of nodes. That is, observing Fig. 6–7–e and f, the number "2" of nodal ordering is attached to a node on the longer edge of the graph, though for Fig. 6–7–a, b, c and d the number is labeled to a node on the shorter edge. It is the difference of the numerical orderings of above cases.
Furthermore, for Fig. 6–7–a, b, c and d,

$$\left[\frac{n}{d+1}\right] \leqslant 3 \qquad (6\text{–}12)$$

But for Fig. 6–7–f,

$$\left[\frac{n}{d+1}\right] \geqslant 3 \qquad (6\text{–}13)$$

That is, the lateral length of the graph in (e) is not enough long to file all of nodes in three nodal rows, while the lengths of graphs in former four cases are long enough.

Here we find another big difference among these six ordering cases. Comparing these results, the former four cases give the optimum state but the last case does not.

Fig. 6-7-g presents a similar form as (e). But they are quite different each other. Under the graph in (g), there exist enough space and the graph must be refiled there and the result is shown in (h). The remapped configuration is exactly same to the graph in (a). And, H. B. W. = 4 for (g). It leads to the conclusion that the graph in (g) is not the final state but only an intermediate one.

From this example in Fig. 6-7, it is guessed that the lines included in the diameter should be displaced along the lateral axis of the filing field as long as possible and it leads to the optimum numerical ordering of nodes.

Paying attention to the additional descriptions in previous example, the reduction method of half bandwidth which is proposed in this section is applied to some actual structural systems and the possibility of the range of application of the method to mesh structures is discussed.

In following examples, the degree of freedom of a node is equated to one for the simplicity, though the degree of freedom is equal to 6 or 3 in actual cases.

[Example 1]. Cable-stayed Bridge

Fig. 6-8 shows the out-line of a cable-stayed bridge which contains 42 nodes.

Treating this example, the difference of the length of structural elements is large and it induces the difficulty of optimum labeling of the joints. For example, a cable connects two nodes which are located distant each other and the distance between the two nodes changes the value in accordance with the selection of the pathes.

But, the number of meshes included in the system is only 6 and the fact helps us to obtain the optimum filing of the graph in the filing field.

As appointed in this section, the graph shows the topology of a bridge and the longitudinal axis of the bridge coincides with the direction of its diameter. That is, the part of the graph which contributes to form the diameter should be filed in a nodal row as long as possible. In this case, a series of lines which presentsthe stiffening girder are to be placed in the field at first.

In order to remove the difficulty of labeling which is occured by the existence of cables, we count the distance of every node from a specified node, i.e. the mid-point of the bridge for this case. "Distance" of a specified node is counted along the shortest path from the mid-point. The maximum value is 13 for this case and it suggests that the graph can be stretched as long as the value in the field, if it is desired.

Furthermore, it is observed that some nodes have the same numerical number. That is, they are at the same distance from the mid-point, and the part of the graph whose nodes have the same distance from centre may be critical from the view point of bandwidth reduction.

Another important information for the filing is the symmetricity of the graph. For this example, the graph of bridge is symmetric with respect to the mid-point of the graph. Furthermore, the half-span of the bridge is almost symmetric with respect to the tower.

By the aid of these informations, the graph is mapped in th filing field without the specification of allowable directions of lines.

The graph is refiled in a new filing field in order to occupy the vacunt places and also to let all lines obey the conditions of directions. The final state is obtained as shown in Fig. 6–8–b. Counting the number of nodal rows, H. B. W. is obtained as follows.

$$\text{H. B. W.} = 5 + 1 = 6.$$

Optimum nodal ordering is given in Fig. 6–8–a.

From this example, it is obtained that the symmetricity of the graph is also very important factor for the reduction procedure of bandwidth. This importance will be appreciated more and more when more complicated systems are treated.

[Example–2]. Cantilever-truss Bridge.

For the example, the author treats "The Harbor Bridge" in Osaka City which is already established. The configuration of the bridge is shown in Fig. 6–9–a and configuration is the graph, itself. For simplicity, it is treated as a plane truss structure and only the side-view is presented.

Though it includes a lot of lines with different length, it is obvious that the longitudinal axis coincides with the diameter. The maximum width across the diameter is easily obtained in a glance and it is equal to three.

But the graph includes a lot of lines inclined not in the allowable directions.

Filing the graph in the filing field with the restrictions of direction, the graph has a configuration like a radder. As the field has enough unoccupied area below the filed graph, it is refiled and the final state is shown in Fig. 6–9–b. Thus,

$$\text{H. B. W.} = 4 + 1 = 5.$$

This structure contains 132 nodes but is very simple one from the view point of bandwidth problem. At first, the difference of length of lines is not so large as the graph in previous example. The second reason is that mesh is composed with only a few lines. The third reason is induced from the configuration, itself. That is, the width at any part of the structure is almost equal to, and the diameter is dominant comparing to other dimensions.

[Example–3]. Loop-shape Bridge.

Fig. 6–10–a shows the model for the analysis which expresses the approach of the Senbonmatsu Bridge in Osaka City. In order to take clearance for cannal, the access lump shows the spiral configuration.

The graph of the structure contains 97 nodes and 103 lines except datumn nodes and the lines which are directly connected to the datumn nodes.

In a glance it seems to be difficult to find out the diameter but careful inspection leads to the conclusion that one end of the diameter is on a node in a lower spiral and another locates on the opposite side in the upper spiral. At P_1 in the model the upper and the lower spirals are not connected each other, but from P_2 to P_6 they are connected by mesh of pier structures.

If they were also free as P_1-position, the diameter of the graph is the total length of the spiral lines.

Thus, the diameter is obtained, at first, by picking up two nodes on the upper and the lower spiral at P_2 and P_5 and followingly by stretching the graph as long as possible, and at the end by adding the length of lower spiral lines up to P_2.

Using this diameter, the graph is filed in the filing field carefully and the final state is shown in Fig. 6-10-b. It shows one of the optimum filed forms and we obtain that

$$\text{H. B. W.} = 8 + 1 = 9.$$

From the filed state, it is obvious that the mapped graph cann't be refiled as to have less number of nodal rows.

This example is rather complex comparing to other two examples in this section. But we could obtain the optimum numerical ordering of nodes by careful treatment of the graph and also by use of the filing field.

6–4. Conclusions

In these investigations, the sequential file method is directly applied to mesh structures, though the method is not developed for them but only for tree structures.

The half bandwidth of some types of mesh structures can be easily obtained in accordance with the procedure for the tree systems, and for the direct application of the sequential file method to mesh systems all parts of the mesh system are treated as nodal sequences in tree system, that is, the mesh graph is transformed to a modified tree graph.

But as far as mesh systems are treated as tree graphs, the maximum degree included in the transformed tree is restricted within a few and the application of Sequential File Method becomes rather simple.

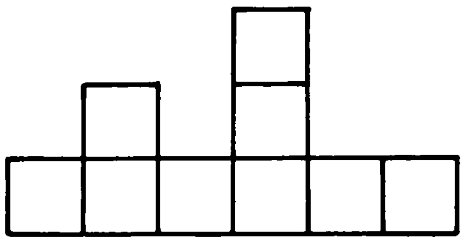
Even if the mesh graph is complicated and the application of sequential file method to the graph seems to be difficult, the optimum numerical ordering of nodes can be obtained only by using the filing field, when the direction and the location of the diameter is obvious. That is, the diameter is the most important factor for finding the minimum bandwidth of the system.

The introduction of the diameter for the nodal labeling may not fail to lead the analyst to minimum half bandwidth and by using the filing field he can check whether the state of numerical ordering is near the optimum one.

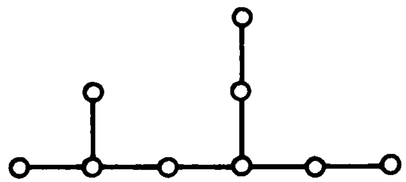
The type of structures which were treated in Section 6 2 is not favourable for the bandwidth reduction algorithms which were proposed in past studies[31-41,45], because they can be treated as a kind of tree graph. Speaking more precisely, the characteristics of these examples are not taken into consideration for the past algorithms and they cann't be treated by the algorithms. The algorithms can effectively treat those whose surrounding shapes are rather simple and are convex, and the examples in this section have concave surrounding configurations.

In the sense, we can conclude that the proposed method for bandwidth reduction is preferable for treating structures with complex surrounding configurations.

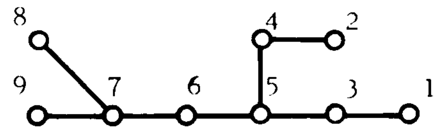
Generally speaking, some types of framed structures have very complicated topology and for some cases the method mentioned above cann't be applied. Thus, further studies must be done and it is hoped that another effective method will be developed.



(a) An Example of Frame

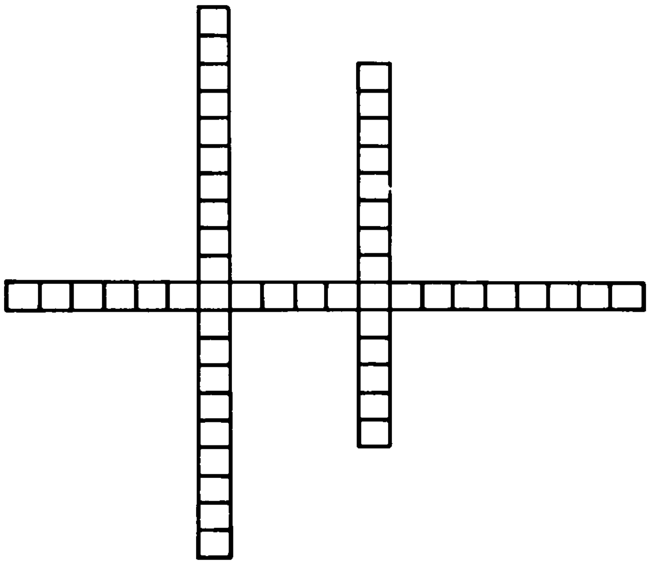


(b) Obtained Tree System

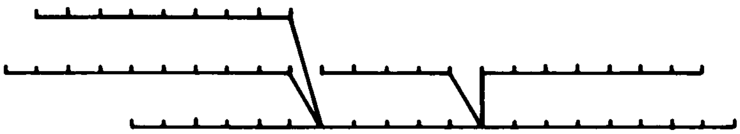


(c) Sequential File Method

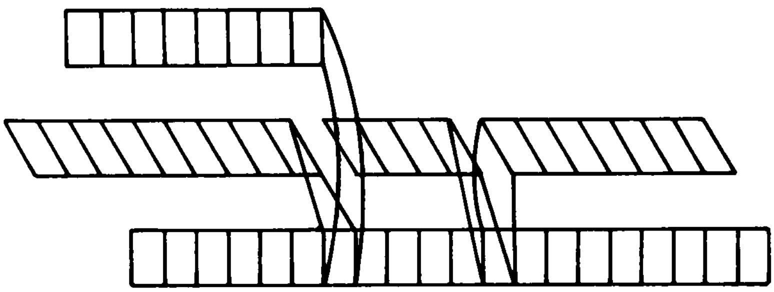
Fig. 6–1 Application of Sequential File Method to Framed Structure



(a) Original Framed Structure with 106 Nodes

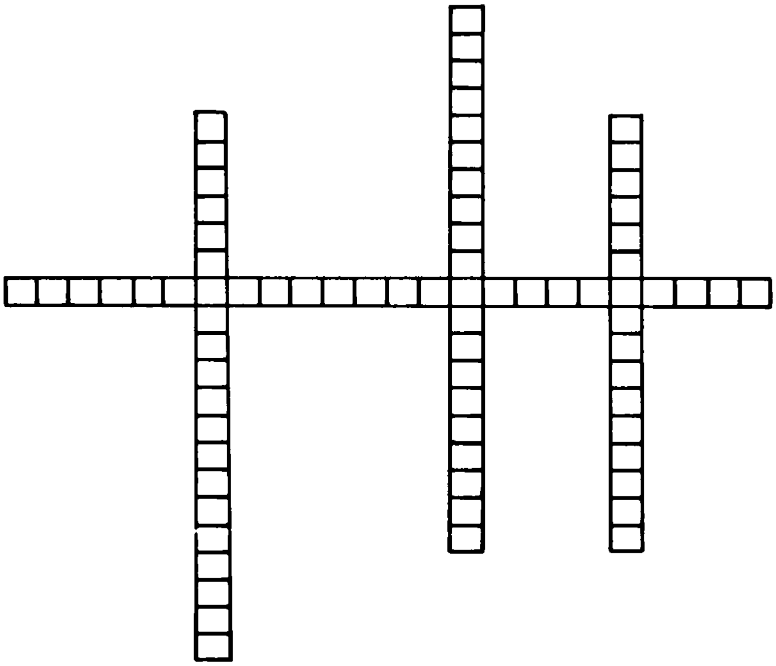


(b) Filed Form of Tree System

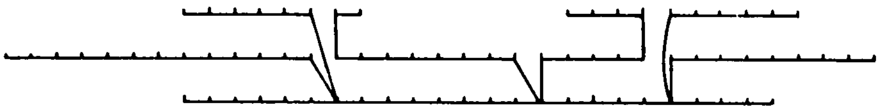


(c) Filed Form of Original Framed System

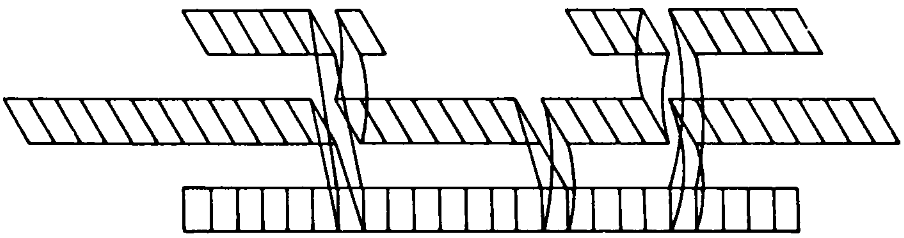
Fig. 6–2 Application of Sequential File Method to Mesh Structure



(a) Original Framed Structure with 156 Nodes

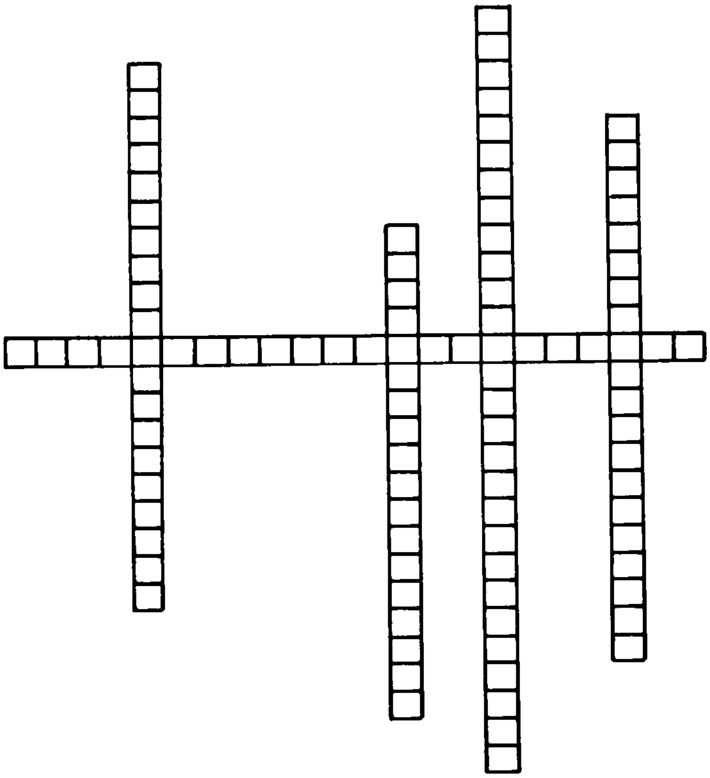


(b) Filed Form of Tree System

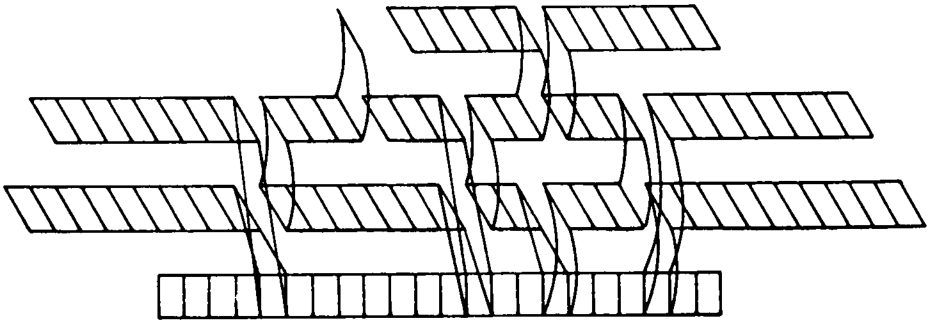


(c) Filed Form of Original Framed System

Fig. 6 3 Application of Sequential File Method to Mesh Structure

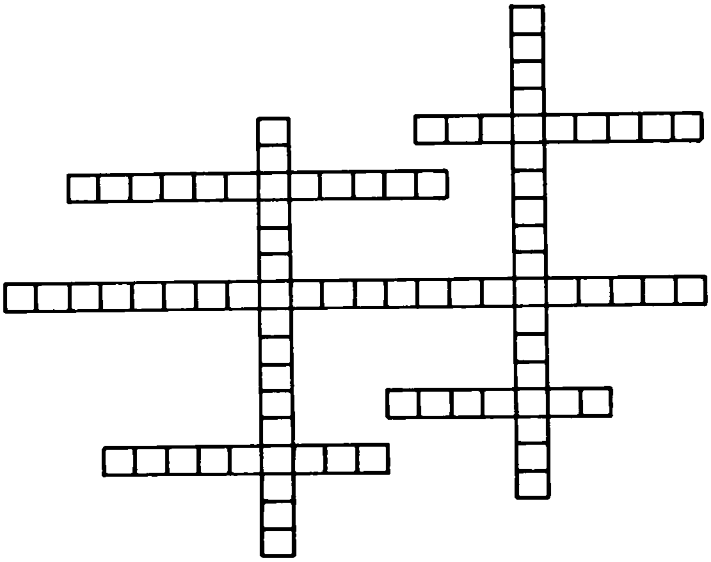


(a) Original Framed Structure with 210 Nodes

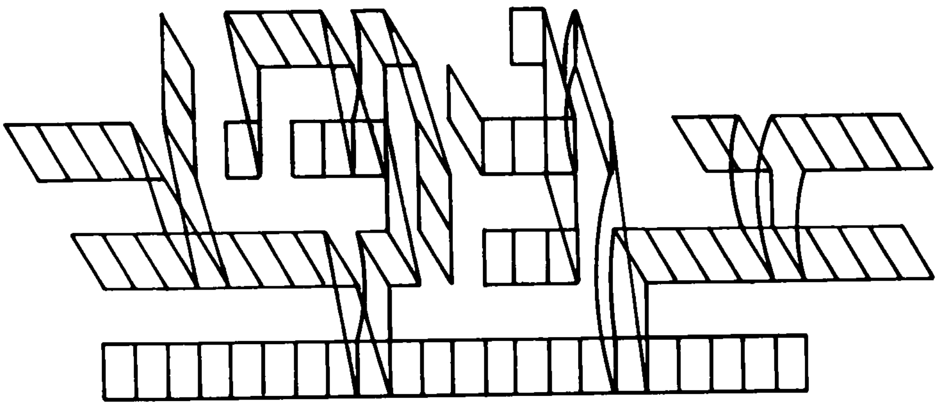


(b) Filed Form of Original Framed System

Fig. 6–4 Application of Sequential File Method to Mesh Structure

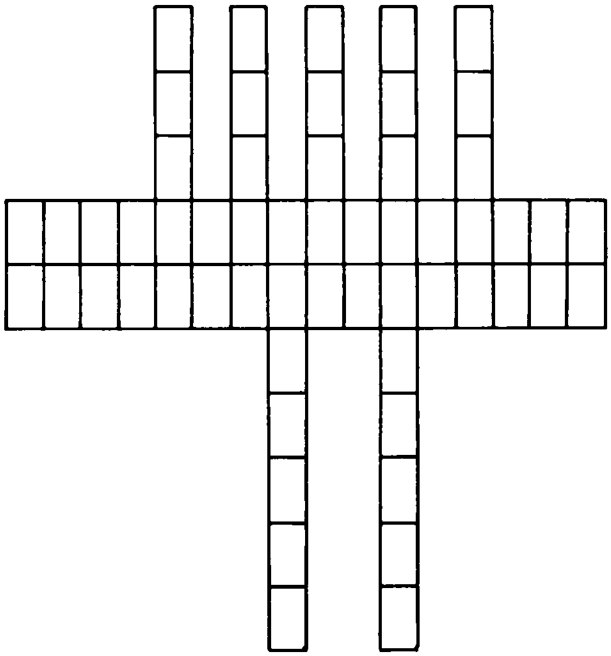


(a) Original Framed Structure with 176 Nodes

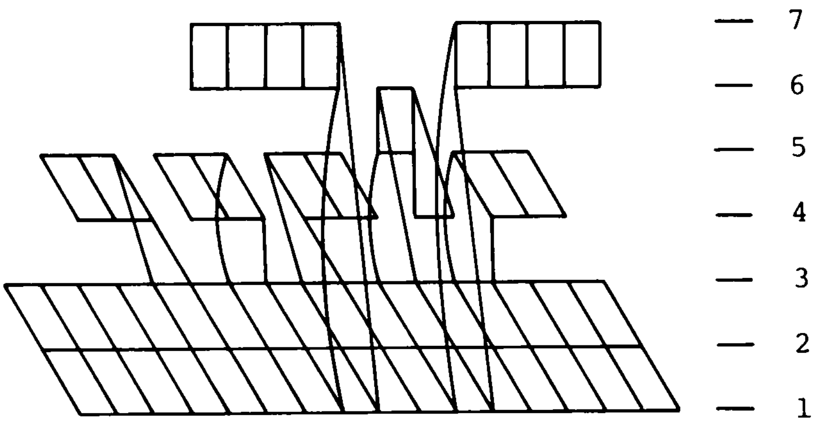


(b) Filed Form of Original Framed System

Fig. 6-5 Application of Sequential File Method to Mesh Structure

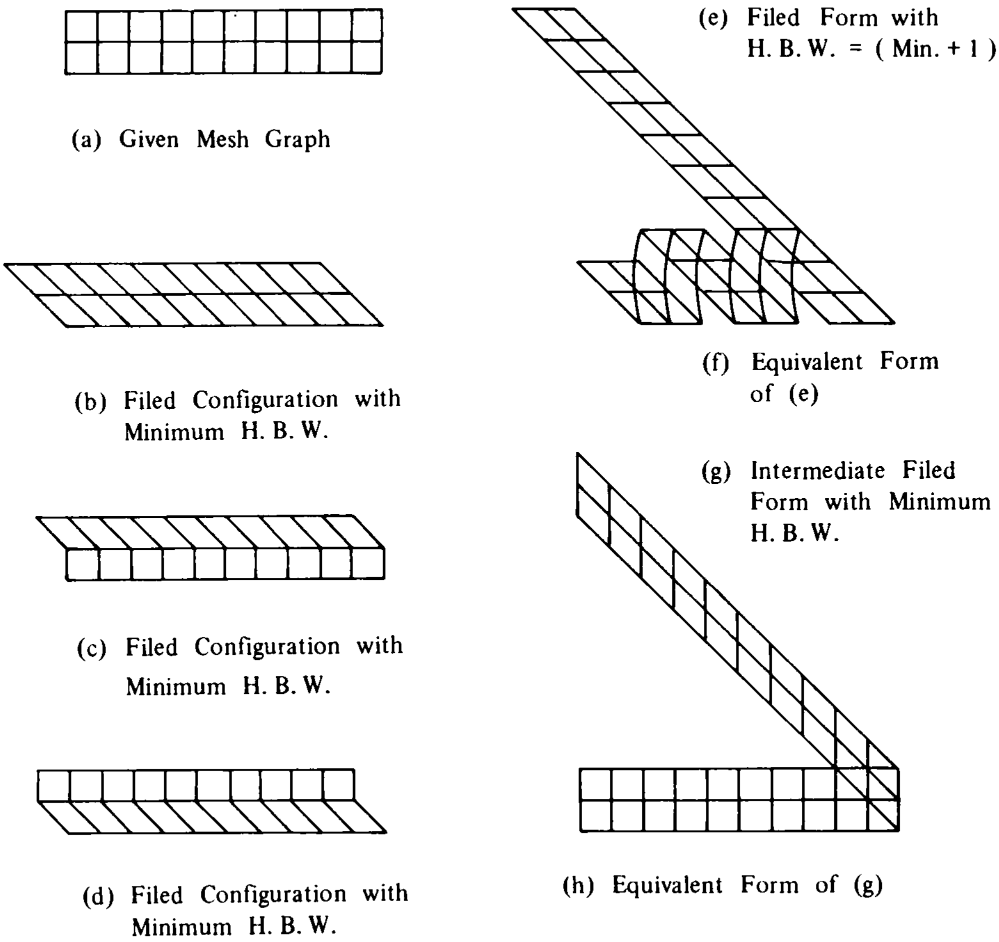


(a) Given Structure with 101 Nodes



(b) Final Configuration in Filing Field; H. B. W. = 8

Fig. 6–6 Direct Application of Sequential File Method to Mesh Structure



(a) Given Mesh Graph

(b) Filed Configuration with Minimum H. B. W.

(c) Filed Configuration with Minimum H. B. W.

(d) Filed Configuration with Minimum H. B. W.

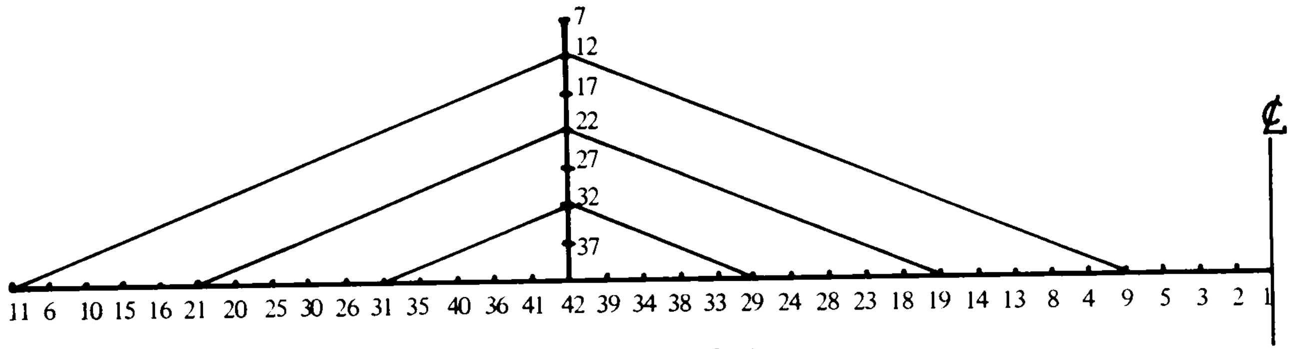
(e) Filed Form with H. B. W. = (Min. + 1)

(f) Equivalent Form of (e)

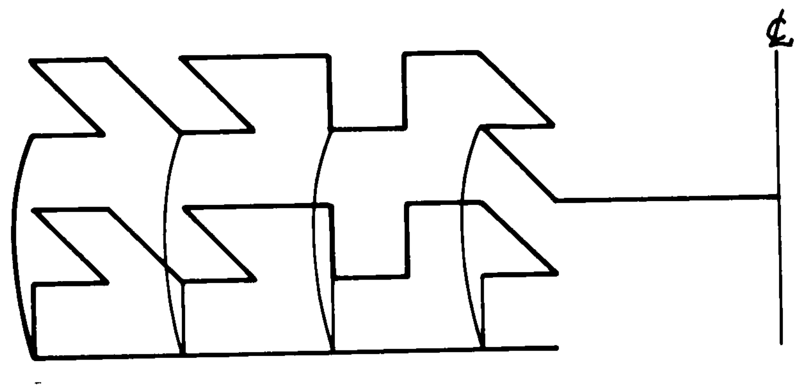
(g) Intermediate Filed Form with Minimum H. B. W.

(h) Equivalent Form of (g)

Fig. 6–7 Various Types of Filed Configurations of A Simple Mesh Structure

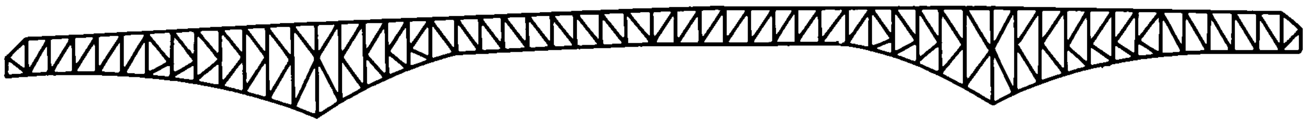


(a) Original Model of Cable-Stayed Bridge

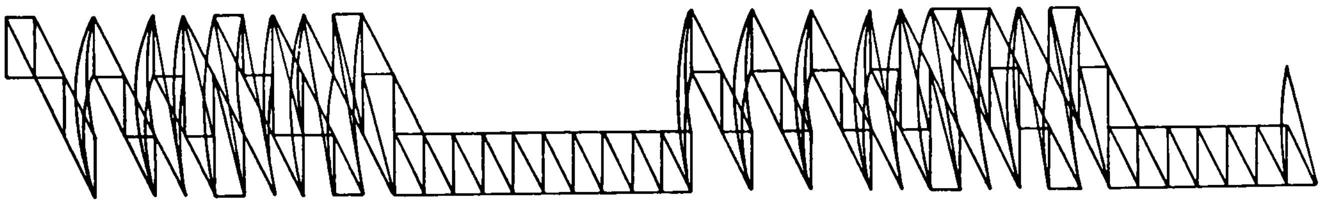


(b) Filed Configuration of The Structure ; H. B. W. = 6

Fig. 6–8 Application of Sequential File Method to A Model of Cable-Stayed Bridge

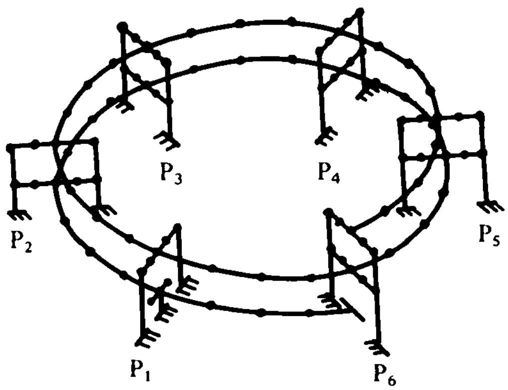


(a) Original Configuration of Truss Type Bridge

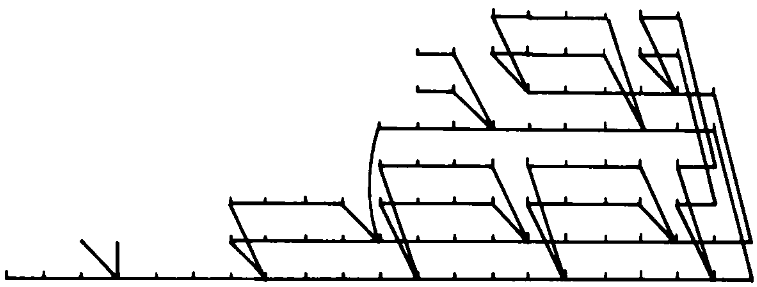


(b) Filed Configuration of The Structure ; H. B. W. = 5

Fig. 6–9 Application of Sequential File Method to Truss Type Structure



(a) Model of Senbonmatsu-Ohashi Bridge



(b) Filed Form of Senbonmatsu Bridge

Fig. 6–10 Application of Sequential File Method to Loop-Shape Viaduct

CHAPTER 7
BANDWIDTH REDUCTION METHOD FOR TWO-DIMENSIONAL CONTINUOUS MEDIA

7-1. Introduction

This chapter contributes to present the bandwidth reduction method for continuous media.

Nowadays, the finite element method is valid for the analysis of continuous media. The method treats a continuum as a gathering of discrete systems and the behaviour of the original system is measured at a number of specified nodes on the boundaries of elements. Through the nodes all of the physical values are transmitted in neighbouring elements. In the sense, the structural properties of a continuous media is transformed to structural ones of an equivalent networks whose nodes correspond to the specified ones and whose lines are imaginarily attached between every neighbouring nodes in an element. Generally, the configuration of an element forms triangular or quadrilateral. For the former case, the configuration of a triangular element includes three nodes and three lines which present the boundary of the element. Thus, the incidence between every neighbouring nodes is taken place by the line in its original model of the element and the graph may be denoted by a complete graph, i.e. $G_c\,(3,\ 3)$. But for the latter case, the configuration of the structural element is insufficient to express the exact incidence between nodes in an element, and two more lines must be attached between every two nodes which locate across the element. Thus, the graph of the model for a quadrilateral element is denoted by a complete graph with four nodes, i.e. $G_c\,(4,\ 6)$. That is, the graph of mechanical property of a finite element must be shown by a complete graph with the same number of nodes as the element has.

In dividing a continuous media, the distribution of nodes on the media is decided by following factors;

1. The loading system
2. The existence of the area where stress concentration is expected.
3. The shape of boundary configration
4. Expected accuracy of the result , and et al.

That is, a part of the media which corresponds to the above items will be divided with smaller-sized elements comparing to the other part. At the same time, the core memory of the machine is limited and also the saving of the computation time is expected. Thus, essentially the non-uniformity of the nodal distribution will appear on the surface of the media.

This non-uniformity of nodal distribution leads to the inequality of the lengths of the boundaries of the elements. In past studies, it is said that the three lengthes of edges for a triangular element should be taken to be approximately equal from the view point of the accuracy of the results. Even if it is held in a part of the media, it cann't be kept for whole area by the reason of the core memory. Thus, inebitably occures the non-uniformity of nodal distribution and also the inequality of the edge lengthes of elements. Therefore, even if the original configuration of the media is very simple, the subdivision of the system may induce the occurence of the difficulties for nodal labeling.

Previous investigations in this thesis contribute to treat a physical system whose boundary is very complicated. But, the above considerations for continuous media teach us that the difficulties in treating the continuous media are different from the difficulties for tree or simple mesh systems.

It is obvious that the difficulty is mainly induced by the difference of the lengthes of edges for each element. Thus, in order to remove it, the concept of "distance" in graph theory should be introduced.[47-51]

As described in Chapter 3, the introduction of the concept induces the modification of the configuration of original system. Saying more exactly, the most important factor which causes the difficulties of treating continuous media is that the direction and the location of the diameter is vague comparing to the others.

Some types of structures which are, of course, divided into discrete system, show their diameters apparently or may give the outline of them on the whole. But, some types hide them and by the mere inspection and also by experience they cann't be obtained.

As described already, the diameter of a system is most important for optimum nodal labeling. Thus, the method to find out the diameter is needed.

The reason why the diameter cann't be searched is caused by the inequality of the edge lengthes between every two nodes in the system.

Comparing two figures in Fig. 7–1, the configurations of the outer boundaries of two plates are the same. (a) shows that a plate is uniformly divided and it contains 45 nodes. On the other hand, (b) shows that the same plate is not divided uniformly and it contains 42 nodes. In the former case, the location and the direction of the diameter are obvious and there are a lot of diameters for the system, and only one of them is shown by a thick line in the figure. But for the latter, the diameter cann't be found by mere inspection, though it includes less number of nodes comparing to the former. Generally speaking, if a system contains less number of nodes, the diameter is obtained more easily. This is not right for this example.

It is obvious that the reason is caused by the inequality of the edge lengthes. Thus, they should be equated each other for the investigation of the diameter.

If we use the transformed configuration, the location and the direction of the diameter may appear more apparently in its figure. That is, as far as the bandwidth reduction is our interest, the actual length of an edge is not needed but the distance in graph theory must take place of it. Thus, the original form should be modified as the new figure can give the analyst the most important factor for labeling on nodes, i.e. the diameter.

Furthermore, the introduction of the three-dimensional filing field is also necessary in order to clarify and also to express the modified configuration.

As described in former chapter, the bandwidth reduction method is changed to how to draw the original configuration with the smallest radius on the three-dimensional filing field.

In the sense, it needs not to equate the lengthes of all edges in the media but only some of them.

For example, a plate may be modified into a curved surface. If a plate has a hole, it may be figured by a cylindrical shell. The original configuration cann't be held, though the topology holds. That is, the number of boundaries is kept in new structure, though the configuration of the boundary will be deformed.

Thus, the author discusses, at first, the details of boundary of original structure and also of the finite elements. Using the results, he shows how to count the number of boundaries of original graph. It will be useful for the classification of the structures.

Following this, the configuration of original boundary and also of the elements in the filing field will be examined by supposing that the system is optimaly mapped. At the same time, the informations for bandwidth reduction are investigated.

Using the results obtained in these sections, a new reduction method of bandwidth is porposed and the merits are discussed and also compared with the sequential file method and other algorithms.

7–2. Boundaries of Continuous Media and Its Elements.

As described in the introduction of this chapter, the use of the three-dimensional filing field changes the configuration of the original system but does not change the number of boundaries. The fact can be said for every element, too

This section contributes to explain the relation between the boundaries of original system and the boundaries of its finite elements, and the author shows how to count the number of original boundaries by use of the boundaries of finite elements.

If a divided area has a triangular form, its boundary coincides with the network-topology of the element. But, if it is a square form, its boundary does not coincide with the topology of the model. What we treat in this section is not the network-topology but the boundary.

Fig. 7–2 shows an example of a plate, denoted by A, which is divided into a number of square structural elements in accordance with the concept of finite element. That is, it is divided into three elements which are denoted by ΔA_1, ΔA_2 and ΔA_3. This example has only one boundary around the original plate.

If we denote the boundary by $\partial(A)$, it is shown by following equation.

$$\partial(A) = a_1 + a_2 + a_3 + a_5 + a_6 + a_8 + a_9 + a_{10} \qquad (7-1)$$

, where ∂ is called "the boundary operator" which is operated in order to reduce the dimensions of the applied structure.[54,55,57] In this example, the application of ∂ to two-dimensional structure induces one-dimensional structure, which is the boundary of the original structure.

The operation of ∂ to every sub-structure gives following results.

$$\begin{aligned} \partial(\Delta A_1) &= a_1 + a_2 + a_3 + a_4 \\ \partial(\Delta A_2) &= a_4 + a_5 + a_6 + a_7 \\ \partial(\Delta A_3) &= a_7 + a_8 + a_9 + a_{10} \end{aligned} \qquad (7\text{–}2)$$

The summation of these results yields to

$$\begin{aligned} &\partial(\Delta A_1) + \partial(\Delta A_2) + (\Delta A_3) \\ &= a_1 + a_2 + a_3 + a_5 + a_6 + a_8 + a_9 + a_{10} \\ &= \partial(A). \qquad (\mathrm{mod}.\ 2) \end{aligned} \qquad (7\text{–}3)$$

The area, A, is the summation of three sub-areas.

$$A = \Delta A_1 + \Delta A_2 + \Delta A_3 \qquad (7\text{–}4)$$

Applying the boundary operator to the above equation, (7–4), following relation is obtained.

$$\begin{aligned} \partial(A) &= \partial(\Delta A_1 + \Delta A_2 + \Delta A_3) \\ &= \partial(\Delta A_1) + \partial(\Delta A_2) + \partial(\Delta A_3) \end{aligned} \qquad (7\text{–}5)$$

This equation suggests that the boundary of the original structure is the summation of the boundaries of the subdivided elements.

Further application of ∂ to the above equation yields to

$$\begin{aligned} \partial(\partial(A)) &= \partial(\partial(\Delta A_1) + \partial(\Delta A_2) + \partial(\Delta A_3)) \\ &= 0 \end{aligned} \qquad (7\text{–}6)$$

That is, the boundary of a boundary is equal to zero. Or, a cycle has no boundary.

Followingly, a structure with two boundaries is also investigated whether the above relation between boundaries of original structure and its subdivided areas is held.

Fig. 7–3 is a plate with a square hole at its centre. The structure is subdivided into 8 elements as shown in the same figure. C_1 and C_2 are two boundaries of area A. Thus,

$$\partial(A) = C_1 + C_2 \qquad (7\text{–}7)$$

Taking the boundary of each subarea, they are obtained as

$$
\begin{aligned}
\partial(\Delta A_1) &= a_1 + a_2 + a_3 + a_4 \\
\partial(\Delta A_2) &= a_3 + a_5 + a_6 + a_7 \\
\partial(\Delta A_3) &= a_6 + a_8 + a_9 + a_{10} \\
\partial(\Delta A_4) &= a_{10} + a_{11} + a_{12} + a_{13} \\
\partial(\Delta A_5) &= a_{13} + a_{14} + a_{15} + a_{16} \\
\partial(\Delta A_6) &= a_{16} + a_{17} + a_{18} + a_{19} \\
\partial(\Delta A_7) &= a_{19} + a_{20} + a_{21} + a_{22} \\
\partial(\Delta A_8) &= a_{22} + a_{23} + a_{24} + a_4
\end{aligned}
\tag{7-8}
$$

Summarizing the boundaries yields to be

$$
\begin{aligned}
\sum_{i=1}^{8} \partial(\Delta A_i) &= \partial(\Delta A_1) + \partial(\Delta A_2) + \partial(\Delta A_3) + \partial(\Delta A_4) \\
&\quad + \partial(\Delta A_5) + \partial(\Delta A_6) + \partial(\Delta A_7) + \partial(\Delta A_8) \\
&= C_1 + C_2 \qquad (\mathrm{mod.}\ 2)
\end{aligned}
\tag{7-9}
$$

Thus, we conclude that following equation exists.

$$
\partial(A) = \sum_{i=1}^{8} \partial(\Delta A_i) \tag{7-10}
$$

That is, the boundary of the original structure is obtained by summarizing the boundaries of the subdivided elements.

Above investigation leads to the conclusion that the boundaries of finite elements include the boundaries of the original system and the latter can be easily found by use of module 2.

The examples are only the elements with square configuration but the procedure is established, even if the configuration is triangular.

The results obtained here do concern with the boundaries of original and its elements but not with the length of them. The bandwidth problem is the problem of topology of structure just treating. And the strategy of the author to obtain a bandwidth reduction method is to clarify the topology of structure by the aid of the filing field. That is, the original structure is mapped into the field which has the minimum radius. Thus, in the procedure, only the topology is held and all of the other properties are removed out.

As far as the topology of a structure is held in above procedure for the reduction method, the relations of boundaries between original and its elements must be kept and the number of boundaries of original structure is equal to the number of boundaries of a new structure in the field.

Furthermore, the equations in this section are established for the new one in the field and the fact suggests that the number of boundaries is counted by use of the above equations even if the whole configuration of the mapped structure is not obvious in the field.

. . . .

Following descriptions are not directly used in this section, but they are useful and available for the understanding of the above explanations about the boundary. .

Fig. 7–4 shows a two-dimensional surface and its boundary. Using the result in above sentences, we have following relation.

$$\partial(A) = l = l_1 + l_2 + l_3 + l_4 + l_5 \tag{7–11}$$

By the application of ∂ to a line element, we obtain both ends of it as its boundaries. This is caused by the behabiour of ∂ which reduces the dimension of an applied object by one. Thus,

$$\begin{aligned} \partial(l_1) &= n_1 + n_2 \\ \partial(l_2) &= n_2 + n_3 \\ \partial(l_3) &= n_3 + n_4 \\ \partial(l_4) &= n_4 + n_5 \\ \partial(l_5) &= n_5 + n_1 \end{aligned} \tag{7–12}$$

The summation of the results yields to

$$\begin{aligned} &\partial(l_1) + \partial(l_2) + \partial(l_3) + \partial(l_4) + \partial(l_5) \\ &= 2(n_1 + n_2 + n_3 + n_4 + n_5) \\ &= 0. \qquad\qquad (\text{mod. } 2) \end{aligned} \tag{7–13}$$

Previous investigation in this section teaches us that the boundaries of a system are equal to the summation of the boundaries of its subdivisions. Thus,

$$\begin{aligned} \partial(l) &= \partial(l_1 + l_2 + l_3 + l_4 + l_5) \\ &= \partial(l_1) + \partial(l_2) + \partial(l_3) + \partial(l_4) + \partial(l_5) \end{aligned} \tag{7–14}$$

then,

$$\partial(\partial(A)) = 0. \tag{7–15}$$

These reductions certificate that the boundary of a boundary of a system is equal to zero.

In Fig. 7–4, "l" presents the boundary of "A" and it is called a bounding cycle. That is, the dividing lines in the continous media correspond to bounding cycles.

7–3. Boundaries of Filing Field and Investigation of Optimaly Mapped Configuration

In this section, the characteristics of boundaries of a configuration in the filing field are investigated.

By some appropriate operations, a continous media is mapped in a three-dimensional filing field which is defined in Chapter 4. Describing more precisely, a mesh graph which corresponds to the characteristic of finite elements of a continuous media is drawn in the field whose radius is aimed to be as small as possible.

If the surface of the mapped configuration in the field is closed, the total configuration has no boundary but it is the boundary, itself.

This type of structure corresponds to closed shell type structures. For example, a cubic tank structure has no boundary, when the thickness of the shell is thin enough to be neglected and any part of it can be modeled by two-dimensional structural element. A baloon is also a good example of the kind of structure.

But, as far as a plate structure with boundary surrounding it is treated, the boundary is also mapped in the field and the image must be the one with open section.

If a plate has two holes, the original configuration has three boundaries and the mapped one has also three open sections in order to keep the original topology. If the holes on a plate are separately located, the open sections of the mapped one must be also separated each other. These open sections on the configuration construct the boundaries of the filing field.

As described in the introduction, the actual shape of the object may not be held after the mapping procedure into the field. Thus, in order to imagine and guess the shape of the mapped one, only the number of boundaries of filing field can suggest the correspondence between the original and its image in the field.

Consider a configuration in the three dimensional filing field. For the simplicity of the investigation, it corresponds to a plate structure without any hole or slit. Then, the configuration has only one boundary.

Fig. 7–5–a is an example of a plate which is already divided into a number of finite elements. Fig. 7–5–b gives the configuration being arbitrarily mapped in the filing field. Two nodes on the boundary may be located by $d = 1$ and the shape of the boundary becomes as shown in (b), that is, the two nodes are combined by an imaginary line and they must be neighbouring.

As described in the definition of three-dimensional filing field, the diameter should be placed along the lateral direction in the field. If it is supposed that (b) gives an optimum state or near optimum image, the diameter may locate as shown in the figure. And, we can confirm that the longest one among all cycles which are obtained by cutting off the configuration into as many as d_0 has the most important effect to decide the bandwidth at the state of the mapped configuration.

Under the assumption, the radius which corresponds to the longest cycle will be near the minimum value. In other words, the shape of (b) has the most slender form comparing to the other mapped configurations.

Thus, it is concluded that the bandwidth reduction method is how to reduce the longest cycle in a mapped configuration. For this example, the longest cycle is a line whose both ends locate on the boundary.

This fact cann't be kept for all cases of optimally mappings for bandwidth reduction. As special cases, the longest cycle may not be found by connecting two nodes on the boundary. Fig. 7–6 shows one of them. In a glance, it is noticed that the longest cycle does not concern with the nodes on the boundary, but it is a true cycle in the original system.

Another example is shown in Fig. 7–7. The boundary appears at one end of the mapped configuration and the longest cycle does not include any node on the boundary or the boundary is the longest cycle, itself, if the boundary is the longest cycle in the optimum configuration.

Generally speaking, as far as a plate structure is treated, the type in Fig. 7–5 is a general form and the types of Fig's 7–6 and 7–7 are scarecely obtained.

The above explanation of the optimum configurations is done from the view point of the boundary of the original structure.

If the boundaries of finite elements of the structure are also taken into consideration in the drawing of the mapped form, the direction of some boundaries may not obey the allowable directions of the field, because only the lines which compose the original boundary are arranged in the field.
Then these lines must be rearranged within the allowable directions by use of the procedure of twisting every neighbouring two cycles till the directions are arranged within allowable angle. By this additional operation to the mapped configuration, the lines which compose the original boundary will not be straight on the surface along the diameter but will incline as steep as the twisting operations.

For example, the shape of Fig. 7–5–b will be twisted as shown in Fig. 7–8. But, the shape of Fig. 7–7 has no effect by twisting, though the internal lines of boundaries may, of course, be rearranged.

Above explanations are done for structures that the original ones have only one boundary, and we call them one-boundary cases.

If a plate has a hole, the mapped configuration is also has two open sections. This type is called "two-boundary case".

In Fig. 7–9, the author shows some typical configurations in optimum states with minimum bandwidth.

They correspond to one- and two-boundary cases. Multi-boundary cases with more than three boundaries are not presented, but they are also imagined and guessed from the configurations presented in Fig. 7–9.

The boundaries in Fig. 7-9 are parallel along the lateral direction and they are considered that the boundaries of finite elements are not taken into consideration.

If the finite element has a quadrilateral configuration, the topology presenting the structural property of the element does not coincide with its boundary but includes two more lines which are placed across the element. Therefore, in the stage of rearrangement of the lines in the allowable directions, these lines have to be taken into consideration.

In the case of triangular finite element, this consideration is not needed at the stage of twisting the cycles.

7–4. Bandwidth Reduction Method for Mesh Graph.

This section contributes for the proposal of a new reduction method of bandwidth for mesh structures.

The structural systems which are treated here are the ones in which the locations of the diameters are not easily obtained by mere inspection. Thus, the first step for the reduction method is to obtain how to find out the diameter. The importance of it for the bandwidth problem is already discussed in former chapters, and for the continuous media the diameter presents the largest number of cycles of the mapped configuration in the three-dimensional filing field. In other words, the path with the maximum length among the original system corresponds to it and the value restricts the lateral length of the configuration in the field.

The second procedure following to the investigation of the diameter is the selection of a number of connected lines which correspond to cycles of the mapped configuration.

In general cases, any cycle can divide the original system into more than two subsystems. It is drawn by connecting a number of nodes in the system. The cycle may be a closed mesh or a number of lines with both ends. The two nodes of the latter case locate on the boundary and they are connected by an imaginary line. Thus, both of them form a true cycle and it is mapped in the three-dimensional filing field.

Our aim of this investigation is how to select the maximum cycles whose lengthes should be reduced as small as possible.

In accordance with the above considerations, the author explains the reduction method of bandwidth for plate-like structure. The proposed method consists of following steps.

[Step-1]. Procedure for investigation of the boundaries of filing field.

This step is not necessary for general finite elements of a plate structure.

In this first step, the shape of the configuration in the filing field is guessed, especially the shape of the boundaries.

Generally speaking, the boundary of a plate may be figured as shown in Fig. 7–5. Supposing the original boundary from the mapped one, the half length of the original boundary is the longest pathes between any pair of nodes. That is, a pair of nodes on the boundary construct the both ends of a diameter. As far as the fact is held, the shape of the mapped structure is supposed to be the one as shown in Fig. 7–5. Thus, if the half length of the boundary spans the diameter, any cycle will include at least two nodes on the boundary, though the cycles at both ends of the mapped configuration may not include two nodes but only one node of the boundary. This is presented in Fig. 7–10 and corresponding connected lines are also shown in Fig. 7–10.

The existence of two nodes on the boundary in any cycle gives very important suggestion for the method of bandwidth reduction and the procedure follows to Step–2.

On the other hand, it is also supposed that a pair of nodes on the boundary may not construct both ends of a diameter. In this case, the boundary cann't span the maximum lateral length of the mapped configuration but it occupies only in a part of the length, and the shape corresponds to Fig's. 7–6 and 7–7.

For the case of Fig. 7–6, a number of cycles include more than two nodes on the boundary and these cycles may give the information as described in the first case of Fig. 7–10. The other cycles don't include the node on the boundary. Thus, we can image the original structure as shown in Fig. 7–11.

For the case of Fig. 7–7, only one cycle can include the nodes on the boundary. That is, the boundary, itself, constructs the cycle and the corresponding configuration of the original is as shown in Fig. 7–12.

The cycle which constructs a boundary of a higher dimensional surface is called "Bounding Cycle".[54,55]

The general characteristic of the second and the third cases is that the boundary cann't construct the diameter.

How to classify the first case from the second and the third cases is described as following.

1. Counting the length of the bounding cycle and also of the cycle which is formed by connecting all the nodes located from the bounding cycle by $d = 1$.
2. Comparing the lengthes of both cycles.

If the former is shorter than the latter, the mapped figure does surely not correspond to Fig. 7–10, but to Fig's. 7–11 or 7–12. But, even if the former is longer than the latter, it is not concluded that the configuration shown in Fig. 7–10 is obtained.

Generally speaking, the total of the finite elements of a plate structure is mapped into the three-dimensional filing field which has the configuration as shown in Fig. 7–10. Following this, the author treats only the configurations which correspond to Fig. 7–10. That is, the half length of the boundary is equal to or near to the diameter, though the value of the diameter is unknown. For the other cases, he will give some considerations later.

[Step–2]. Selection of an imaginary cycle across the original graph.

Select a node on the boundary of original structure and denote it as the i-th node. The shortest path from the i-th node to another node (denoted by the j-th node) on the boundary is calculated.

If the j-th node is selected from the right-side one neighbouring to the i-th one and followed to the counter-clockwise direction, the values of the shortest pathes change as presented in Fig. 7-13. Denote d_{ij} the value of the shortest path from the i-th to the j-th node. d_{ij} increases at first and after the reaching to a maximum value it decreases to a minimum value. And it increases again. The minimum value of d_{ij} is denoted by d_i. This change of d_{ij} is the general case, but there often occures another case. That is, d_{ij} increases at first and after taking a maximum value of d_{ij} it decreases without increasing again. In this case we cann't define the minimum value of d_{ij}. This change of d_{ij} suggests the relation between the configuration and the location of the i-th node as shown in Fig. 7–14. That d_{ij} has the maximum value indicates that the path, d_{ij}, gives the diameter of the graph. For this case, we consider that

$$\text{min. of } d_{ij} = \text{max. of } d_{ij} = d_i \tag{7–16}$$

d_i is calculated for every node on the boundary and it gives following sequence.

$$\{ d_1, d_2, \ldots, d_i, \ldots, d_{\bar{n}} \} \tag{7–17}$$

, where $\bar{n}$ indicates the number of nodes on the boundary.

Among the values in the sequence, the minimum value, d_i, is selected and the value indicates the number of nodes in the temporary cycle. On the contrary, the maximum value in the sequence gives the number of nodes which are included in the diameter of the graph, and the path between the i-th and the j-th nodes constructs the diameter.

$d_{ij} = \text{max.}$ is the longest one within the shortest pathes which cross the configuration, and $d_{ij} = \text{min.}$ is the shortest one within the shortest pathes which cross the configuration. If the former corresponds to the length of the configuration along the longitudinal axis, the latter corresponds to the width of the configuration which crosses the longitudinal axis.

Paying attention to $d_{ij} = \text{min.}$, it often occures that the number of nodes which gives the minimum value of d_{ij} is not restricted to one node but there exist, at least, two nodes. Or, it may be general that a number of nodes satisfy the condition of $d_{ij} = \text{min.}$. That is, there are more than one path with minimum width across the configuration. All of the pathes are the candidates of the temporary cycle.

These shortest pathes in the original structural system may construct cycles of the configuration in the three-dimensional filing field, and the actual cycle at the optimaly mapped configuration has, at least, the same number of nodes which are included in $d_{ij} = \text{min.}$. In the sense, the author calls the path with $d_{ij} = \text{min.}$ "temporary cycle". It is evident that the actual cycle in the optimaly mapped state has equal to or more nodes than the temporary cycle has.

Generally, at the procedure to obtain d_{ij} in which "i" is fixed, the change of values of d_{ij} may be more complicated than the description in above sentences. That is, the decreasing of value may occure more than two times, though the case with only one time decreasing is described above.

These general cases must be studied in future and in this section they are not treated. But saying from the author's experiences, d_{ij} which is obtained in this step can give rather good result for the bandwidth reduction. – It is caused by the reason that the cycle with $d_{ij} = \min.$ is an temporary cycle and at the following steps for obtaining the actual cycle from the temporary one the number of nodes included in the cycle is increased to a certain extent.

[Step–3]. Procedure for making actual cycles in the three dimensional filing field.

One of the shortest pathes across the original configuration is drawn by connecting two nodes on the boundary. A number of nodes of the interior part of the structure are, of course, included in the path.

In Fig. 7–15, the shortest path is denoted by d_i, which includes d_i nodes. Thus, (d_i-2) nodes are from the interior area and two nodes are the nodes on the boundary.

The temporary cycle is, at first, taken as the actual cycle in the system. The other cycles are constructed as follows.

1). The nodes on the right side of the temporary cycle, i.e. d_i, which are connected to the cycle by $d = 1$ are selected and the number of nodes are counted.

2). If the number of nodes is less than d_i, a new cycle, d_{i-1}, is constructed by connecting them all. If d_{i-1} is larger than d_i by more than two, some of them in d_{i-1} are selected and added to d_i. Thus, a new cycle, d_i, is constructed and at the same time neighbouring new cycle, d_{i-1}, is formed by the above procedure. Repeating the procedure, we obtain a sequence of cycles, i.e.

$$(d_i, d_{i-1}, \ldots , d_3, d_2, d_1) \tag{7-18}$$

for the right side of d_i.

3). The same procedure is repeated for the left side of d_i and we obtain the sequence.

$$(d_{\bar{n}}, d_{\bar{n}-1}, d_{\bar{n}-2}, \ldots , d_{i+1}, d_i) \tag{7-19}$$

4). Any node in the configuration is included in a cycle and we obtain the sequence.

$$C_i : (d_1, d_2, \ldots , d_{i-1}, d_i, d_{i+1}, \ldots , d_{\bar{n}-1}, d_{\bar{n}}) \tag{7-20}$$

C_1 indicates a sequence of cycles for case 1.

5). Draw the cycles in the three-dimensional filing field and give the original connectivity between every neighbouring cycles.

If there exist lines which don't obey the restrictions of allowable direction, the left side cycle is twisted counter-clockwisely to the right side cycle till all of the lines are rearranged in the allowable directions. And we count the difference of nodal rows in which the i-th and the (i + 1)-th nodes on the boundary locate, and we obtain following sequence of number for twisting of cycles.

$$T_1 : (0, t_2, t_3, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_{\bar{n}-1}, t_{\bar{n}}) \tag{7-21}$$

, in which t_i indicates the difference of twisting between d_{i-1} and d_i cycles.
6). Add C_1 and T_1, and obtain the maximum value of $(C_1 + T_1)$ sequence. The maximum value in the sequence is denoted by $(C_1 + T_1)_{max.}$.

$$(C_1 + T_1)_{max.} = \max_{i=1}^{\bar{n}}. \text{ of } (d_i + t_i) \tag{7-22}$$

7). The procedures from 1) to 6) are repeated for cases of the minimum path, d_i. If there exist α cases, we obtain

$$[(C_1 + T_1)_{max.}, (C_2 + T_2)_{max.}, \ldots, (C_i + T_i)_{max.}, \ldots, (C_\alpha + T_\alpha)_{max.}] \tag{7-23}$$

8). Compare and find the minimum value among above sequence.
The mapped configuration corresponding to the minimum value is the optimaly mapped configuration. And it gives the minimum bandwidth, i.e. H. B. W., for the structure.

$$\text{H. B. W.} = \min_{i=1}^{\alpha}. \text{ of } (C_i + T_i)_{max.} \tag{7-24}$$

The general procedure to find the minimum bandwidth of a plate structure with finite elements consists of three steps above mentioned.

Actual treatment of the procedure is easier than the treatment of abstract object as presented in this section. Some of these procedures may become automatically useless, and some other useful informations to obtain the actual cycles can be added to the proposed method. For example, the symmetricity of the configuration is one of them.

Step–2 and Step–3 are not for general cases but for a case with only one boundary. That is, the steps can be treated only for the configurations which are imagined in Fig. 7–9.

If the original has two boundaries, the optimum state of the mapped configurations are shown in Fig. 7–9.

Among two-boundary cases, (a)-type shows that the boundaries locate only at the both ends of the mapped configuration and they cann't span the diameter. Among (b)-type forms, the first and the second configurations present that one of the boundaries span the diameter, and for the third half length of the boundaries is shorter than its diameter.

Observing these configurations, (b – 1) and (b – 2) types of two-boundary case correspond to the (b – 1) and (b – 2) types of one boundary case which is already treated in this section. Therefore, these types can be treated by the similar procedure for one-boundary case.

But, we have to know which boundary spans the diameter. For the purpose, the lengthes of two boundaries are compared and the longer boundary does surely span the diameter. Following this preparatory work, the procedure in this section is applied for this case.

Note that arbitrary two nodes in the shorter boundary can be connected by an imaginary line by d = 1, as shown in the figures in Fig. 7–9. Taking account of this fact, the temporary cycle for the configuration with two boundaries is obtained. Using the result of temporary one, the actual cycles are successively obtained by the try-and-error method.

Thus, it may be concluded that the procedures proposed in this section is available for two-boundary cases as far as one of the boundaries spans the length of the diameter.

As described in this section, (b – 1) and (b – 2) types of the optimum configuration are general types and the other types are scarecely met for actual structural systems. Here, the author gives some considerations for these exceptional types.

The original nodal distribution which corresponds to (a) and (b – 3) types is not necessarily unifrom but nodes are concentrated in a part of the structural surface. These area corresponds to the left side end of the configurations of (a) and (b – 3) types.

For (a) type, the boundary becomes the actual cycle, itself, and it suggests that the bounding cycle may be chosen for the temporary cycle at the beginning.

For (b – 3) type, only a part of the diameter is covered by the original boundary. Thus, for this part, at least, two nodes on the boundary may be included in a cycle. This fact gives the important information for the bandwidth reduction method. That is, for the area the proposed method to obtain the temporary and real cycles can be applied.

Summarizing these configurations for various types of structures the bandwidth reduction method in this chapter is realized its validity for any type of structural system.

7–5. Application of the Bandwidth Reduction Method to Actual Structures.

This section contributes to show some examples of the application of the bandwidth reduction method proposed in previous section to some actual structures with one or two boundaries.

[Example–1]

Fig. 7–16 shows a very simple example of a framed structure. This structure has no boundary and it includes 18 nodes.

The nodal distribution is uniform and the total length of the surrounding cycle is the longest one. Thus, (b – 1) or (b – 2) type is expected to be appeared in the three dimensional filing field.

Fig. 7–16–a gives the values of $d_{ij=min.}$.

$$\text{Min. of } (d_{ij=min.}) = 5$$

The minimum value appears at four nodes on the surrounding. Taking account of the symmetricity of the configuration, we may choose only one node of them which is marked in the figure. The distance from the representative node to the other nodes on the boundary are shown in Fig. 7–16–b. The minimum value is equal to 2 and the line with d = 2 is selected as the temporary cycle. For this case, the temporary cycle is equivalent to the actual cycle, and the configuration in the three-dimensional filing field is presented in Fig. 7–16–c. The optimum numerical ordering is given in Fig. 7–16–d. The result is one of the optimum and we obtain that

$$\text{H. B. W.} = 3 + 1 = 4$$

, if the degree of freedom of a node is supposed to be one for the simplicity. This assumption is used in following examples, too.

[Example–2]

If the same configuration in example 1 is a plate-like structure, the topology changes as shown in Fig. 7–17–a. The structure contains true boundary and the graph has 18 nodes. For this graph, (b – 1) or (b – 2) type of mapped configuration is expected. Fig. 7–17–a shows the values of $d_{ij=min.}$.

$$\text{Min. of } (d_{ij=min.}) = 3$$

As the symmetricity of the graph is held, we may select only one node among four nodes as a representative one. Fig. 7–17–b presents the distances from the representative node to the other on the boundary. Taking account of the symmetricity, the temporary cycle is selected. Using it, actual cycles are selected. After filing all of the cycles in the field, the twisting operaion is given between every neighbouring cycles and the last state is obtained as shown in Fig. 7–17–c. For this case,

$$\text{H. B. W.} = 3 + 1 + 1 = 5$$

Comparing the result with the one of example 1, H. B. W. increases by one which is induced by the twisting. The additional twisting operation induces the twisted configuration of the original structure.

[Example–3]

Fig. 7–18 shows a framed structure with 20 nodes on its boundary. It is obvious that the length of the surrounding is the longest and it is expected that (b – 1) or (b – 2) type can sufficiently show the optimum state in the filing field.

Fig. 7–18–a presents the value of $d_{ij=min.}$.

$$\text{Min. of } (d_{ij=min.}) = 6$$

Two nodes satisfy the value and they are denoted by (I) and (II). Fig's. 7–18–b and –c show the distance from (I) and (II) to the other nodes on the surrounding, respectively.

$$\text{Min. of } (d_{ij=min.}) = 3 \quad \text{for (I)}$$

$$\text{Min. of } (d_{ij=\min.}) = 2 \quad \text{for (II)}$$

Observing the case of (II), we find a part of structure having the width with more than 2. Thus, the length of the temporary cycle should be taken to be equal to 3. Trial and error method for selecting the actual cycles leads to the result that any cycle has, at least, four nodes. The cycles include (1, 2, 3, 4, 4, 4, 3, 3, 3, 2, 1) nodes when they are successively presented from the left to the right. Twisting is appeared in a part of it, but it does not influence the value of H. B. W..

$$\text{H. B. W.} = 5$$

[Example–4]

Fig. 7–19 shows an example of a plate divided into finite elements. The plate has 26 elements and 40 joints.

(a) presents the connectivity relations between these joints and (b) gives the optimum filed configuration in the three-dimensional filing field.

Comparing the figure with the one in Example 2, the inclination of the original boundary in the field is quite different, because the configuration is drawn on the inner surface of the filing field.

Furthermore, the largeness of the radius of the mapped shape changes in accordance with the number of nodes in cycles.

From this configuration in the field, we know that

$$\text{H. B. W.} = 7$$

[Example–5]

A plate structure with a hole shown in Fig. 3–7 is also treated as an example.

Original configuration is transformed into a structure as shown in (c). And the inspection of the transformed configuration teaches that it has two boundaries and the configuration is refiled into (b)-type in Fig. 7–9, because the both boundaries may span the diameter. (a)-type configuration is not at all supposed from the fact.

From this inspection for the optimum configuration we can easily obtain the minimum half bandwidth.

$$\text{H. B. W.} = 11.$$

[Example–6]

Fig. 7–20 shows an example treating a plate with finite elements. This example is also one boundary case.

The nodal distribution is not uniform and to obtain the optimum numerical ordering is very difficult.

The figure contains 99 nodes, and 33 nodes among them locate on the boundary.

Our experience suggests that the direction of the diameter may be along the line connecting B and E in the figure. Furthermore, the nodal distribution is concentrated in the left side of the system. These facts teach us that the calculation of the shortest path connecting two nodes on the boundary should be done only for a number of them on the boundary.

Fig. 7–20–a gives the values of $d_{ij=min.}$, and we obtain that

$$\text{Min. of } (d_{ij=min.}) = 9$$

This value appears at two nodes and they are denoted by (I) and (II).

The shortest pathes from (I) and (II) to the other nodes are calculated and they are shown in Fig. 7–20–b and –c, respectively.

Both of them show that the temporary cycle has, at least, seven nodes, and there exist more than two nodes which correspond to the value, i.e. 7.

For case (I), four nodes on A–E satisfy the value, i.e. 7, and for case (II), five nodes correspond to the condition.

Taking some trials to make actual cycles, it is realized that the value, i.e. 7, is too small and the value is added by one. Successive trials to cover all nodes by a number of actual cycles with minimum number of nodes reach to a result which is shown in Fig. 7–20–b.

The figure presents the actual cycles in the original system, and the number of nodes in a cycle is, at most, equal to 10. And it is realized that (I) and (II) cases are satisfied in the figure.

Fig. 7–20–e presents the graph in the two-dimensional filing field in which the original is mapped within 10 nodal rows. Fig. 7–20–d and –e are equivalent each other.

In order to rearrange the directions of lines which are not in the allowable directions, two more nodal rows are necessary. This coincides with the twisting operation in the three-dimensional filing field.

Therefore, the minimum half bandwidth is obtained by adding this value for each nodal column and by comparing them each other. And we obtain that

$$\text{H. B. W.} = 10 + 2 + 1 = 13$$

That is, all of the non-zero elements of the stiffness matrix of Fig. 7–20–a are surely rearranged within 13 lines including the main diagonal line. The graph has 99 nodes and 257 connecting lines. Thus, the total of non-zero elements is equal to 356.

The ratio of the number of non-zero elements to the total elements of the matrix is obtained as follow.

$$\rho = \frac{365}{99 \times 99} \times 100 = 3.36 \ (\%)$$

On the other hand, if we use the above result, the ratio of non-zeros to the elements within the value of H. B. W. is given by ρ'

$$\rho' = \frac{356}{1209} \times 100 = 29.44\ (\%)$$

Comparing ρ' with ρ, the ratio of non-zero elements in the stiffness matrix being used is increased about eight times, if we use the proposed bandwidth reduction method. The total number of elements for the band matrix method is only 1209, though the matrix of the stiffness contains 9801 elements.

[Example−7]

Fig. 7−21 shows the same plate as shown in Fig. 7−20 but it is devided differently. The nodal distribution is not uniform and the area near E node of the plate has concentrated nodal distribution.

Any element has a triangular configuration and the configuration of Fig. 7−21−a gives the topology of the graph.

Fig. 7−21−a gives the values of $d_{ij=min.}$. And we obtain that

$$\text{Min. of } (d_{ij=min.}) = 9$$

This value is satisfied at two nodes which are denoted by (I) and (II).

Fig. 7−21−b and −c present the distance from (I) and (II) to any node on the boundary, respectively. Both of them show that the temporary cycle will contain 6 nodes. Thus, the actual cycles contain, at least, 6 nodes.

Using the temporary cycle the procedure to obtain the actual cycles is done and the final result is shown in Fig. 7−21−d.

The maximum number of nodes in any cycle is restricted to be equal to or less than 9.

Comparing every neighbouring cycles, it is obtained that additional one nodal row is enough to rearrange all of the lines in the allowable directions. Thus, the half bandwidth can be calculated as following.

$$\text{H. B. W.} = 9 + 1 + 1 = 11.$$

For the generality, we should express as following.

$$\text{H. B. W.} \leqslant 11$$

Total number of non-zero elements in stiffness matrix is 379. If they are gathered in the half bandwidth which is obtained here, the ratio of non-zeros to total elements in H. B. W. is easily calculated.

$$\rho' = \frac{379}{1111} \times 100 = 34.11\ (\%)$$

Core memory is needed as the number of the total elements in H. B. W., i.e. 1111.

On the other hand, if the calculation is operated for the total stiffness matrix, a computer is required to prepare the core memories of 11,236. And the ratio, ρ, decreases.

$$\rho = \frac{379}{11236} \times 100 = 3.37\ (\%)$$

For this example, the above result of H. B. W. is compared with the results by another method which is usually used for numerical ordering.[38]

This ordinary bandwidth reduction method is summarized as following.

1. Searching a node on the boundary which has minimum degree.
2. The node which is selected in Step 1 is denoted as Unit 1.
3. Searching nodes which are connected to Unit 1 by d = 1.
   These nodes are called Unit 2.
4. Repetition of Step 3 classifies all nodes into as many number of units as the maximum distance from Unit 1.
5. The node in Unit 1 is denoted as "1".
   Giving the numerical ordering to the nodes in Unit 2, two nodes on the boundary are given the initial and the final number in the group and the other are labeled in accordance with the direction from the initial to final node.
6. Step 5 is repeated for every unit.

These procedures are applied to the above example. For Unit 1, A, B, C, D and E nodes are selected. Thus, we obtain five cases for nodal labeling. The results of dividing into units are presented in Fig. 7–21–e, –f, –g, –h and –i, respectively. The best result among these five cases is obtained when A-node is classified into Unit 1, and the maximum number of nodes in any unit is equal to 11. Thus, we can conclude that the half bandwidth is, at best, decreased to 12.

$$\text{H. B. W.} \geqslant 12$$

Comparing the result with the one by use of the proposed method, the ordinary method cann't lead to the minimum result.

For some types of structures, it can give the best results. But the above application shows that the method is not a general method being applicable for any structures.

Whether it can lead to the best or not does wholly depend upon the configuration of the system. The first reason why the method cann't give the minimum value of the bandwidth is that it can select only one node for Unit 1. Another reason is that every two neighbouring units are selected by the condition of d = 1.

Using the proposed method in this chapter, the procedure of division of all nodes into a number of groups does not begin from one end of the configuration but from the central area of it. And it induces the selection of some nodes for Unit 1.

[Example- 8]

Example−8 treats the case with two boundaries. Fig. 7−22−a presents a plate with a hole and the irregular outer boundary configuration. It includes 134 nodes and 339 lines. The nodal distribution is uniform in all area of the configuration.

Comparison of the lengthes of two boundaries leads to the conclusion that the outer boundary is evidently longer than the interior boundary. Furthermore, it is known that the outer one may span the diameter. Thus, it can be said that this configuration corresponds to (b−1) or (b−2) type among the optimum forms, if it is mapped into the three-dimensional filing field.

The inspection of the whole configuration and our experiences teach us that the calculation of $d_{ij=min.}$ may be restricted only in a part of the whole area.

Fig. 7−22−a shows the values of $d_{ij=min.}$ for a number of nodes on the boundary. Then, it is obtained that

$$\text{Min. of } (d_{ij=min.}) = 10$$

This value is satisfied at two nodes. As a representative node, the node denoted by (I) in Fig. 7−22−a is selected.

Fig. 7−22−b presents the value of the shortest path from the node to an arbitrary node on the same boundary. In the calculation of the length of a path, arbitrary two nodes on the interior boundary can be connected by d = 1. This procedure corresponds to giving an imaginary line connecting two nodes on the boundary. Minimum path length across the graph is equal to 4 and the value is observed at two nodes.

Inspection of the whole configuration finds that the graph is almost symmetric with respect to an axis given in Fig. 7−22−a. Taking account of the symmetricity of the graph, the upper node among these two which are selected to construct one end of a temporary cycle is finally taken as the representative node for the temporary cycle.

Fig. 7−22−c shows an example of the actual cycles which are directly produced by use of the temporary cycle. The result obtained in the figure is rather bad, because the inclination of the actual cycles which locate in the lower part of the graph presents shallow angle with respect to the lateral direction of the graph and they include much nodes. That is, the symmetricity of the graph is not appeared in the area when the graph is divided into cycles as shown in Fig. 7−22−c. If the symmetricity is held in producing the actual cycles, the cycles in Fig. 7−22−c are modified and it yields to Fig. 7−22−d.

In the new figure, the actual cycles are almost symmetrically placed in the graph, and the maximum number of nodes in any cycle is equal to 10. It means that any cycle contains, at most, 10 nodes. From the value we can guess that the half bandwidth of the graph must be larger than 11.

$$\text{H. B. W.} \geqslant 10 + 1 = 11.$$

The strict value of H. B. W. is equal to 13, that is, the twisting operation must be applied to the graph in order to rearrange the inclination of lines in allowable directions.

ρ and ρ' for this example are presented as followings.

$$\rho = \frac{473}{18056} \times 100 = 2.62 \ (\%)$$

$$\rho' = \frac{473}{1664} \times 100 = 28.42 \ (\%)$$

Using the band matrix method, only 1664 core memories are needed, though the stiffness matrix includes 18056 elements in it.

[Example–9]

This example is also a plate with a hole as shown in Fig. 7–23–a. It has 171 nodes and 451 lines.

The nodal distribution is not uniform but is concentrated in the area near the hole. But it seems to be almost symmetric with respect to the axis shown in the figure.

In Fig. 7–23–a, it is shown that the minimum value of the longest pathes between nodes on the outer boundary is equal to 13.

Selecting a node which satisfies the value as a representative node, the distance of both ends of the shortest pathes across the graph is equal to 6, as shown in Fig. 7–23–b. The node which is apart from (I) by d = 6 locates rather lower part of the graph, and the actual cycles in accordance with the value are shown in Fig. 7–23–c. The result shows that the half bandwidth is, at least, larger than 20 for the case.

If the symmetricity of the graph is taken into account for producing the actual cycles, the value is to be reduced and is equal to 7. Thus, the temporary cycle contains 8 nodes. Using this cycle, the actual cycles are sought by trial and error method. Fig. 7–23–d presents the final actual cycles, one of which includes, at most, 13 nodes. Thus,

$$\text{H. B. W.} \geqslant 13 + 1 = 14$$

If the ordinary method for nodal labeling is applied to this graph, one of the results is shown in Fig. 7–23–e, in which the maximum number of nodes in any cycle is equal to 18.

$$\text{H. B. W.} \geqslant 18 + 1 = 19$$

This result is, of course, worse than the one which is obtained by the author's method.

7–6. Conclusions

In this investigations the author proposed a bandwidth reduction method which can be applied for plate-like structure that are subdivided into triangular or quadrilateral finite elements. By use of the three-dimensional filing field the method becomes valid.

In the stage for the classification of the mapped configurations in the field, it is realized that the number of boundaries of original systems is the most important, and the author gives the method how to count the number of boundaries. And also, the types of mapped configurations are classified in accordance with the number of boundaries.

The direct application of the proposed bandwidth reduction method may lead to the minimum value of the bandwidth.

The characteristic of the method comparing to the others is that the decision of cycles is done from the central area of a system, though the others decide, at first, a cycle which locates at one end of the mapped configuration. Furthermore, before actual application of the method to a system, the inspection and the imagination of the optimum mapped configuration are done by the aid of the diameter, the number of boundaries and the filing field.

It is true that the forecast of the optimum state may require rich experiences of the nodal labeling. But, as far as the direction and the location of the diameter are taken into consideration, the optimization procedure can be forced to be proceeded and the final result can be obtained. The forecasting can save the work which is used, especially, in the step to obtain actual cycles.

The most important factor to obtain the minimum bandwidth is the forecasting of the final state. As far as the diameter is surely taken into consideration, we may not fail to lead to the worse result for any type of structural system.

The bandwidth reduction methods proposed up-to-days have no inspection and the imagination of the final state. It is caused by the reason that they are not graphical ones but they are introduced only for the use of the digital computers.

Generally speaking, only a part of a structural system influences the value of the bandwidth. Thus, in order to reduce the value to the minimum, the part should be treated carefully.

But, as far as the usual methods are applied, they treat a system from one end to another. Thus, the most important area may be influenced by the treatment of the unimportant area. Therefore, we can conclude that whether they can give good results depends on the configuration of the system.

By use of the proposed method the influence of the configuration to the final result is, at least, decreased, because the central area of the system is, at first, treated. Therefore, as far as the configuration is ordinary type as shown in examples in this chapter, the proposed method is valid.

But, if it has the configuration as shown in Fig. 7–24, that is, the area of both ends are wide and the central area is comparatively narrow, the direct application of the proposed method is not useful, though it can induce the optimum result at the end of repitition of the proposed procedure.

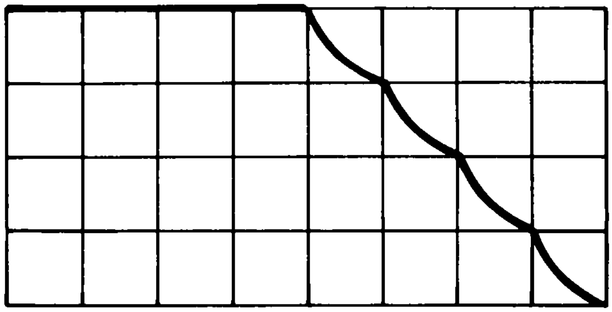
By the direct application, the obtained temporary cycle contains very few number of nodes comparing with the other part of the graph. That is, the initial cycle is far different from the critical actual cycle which appears at both ends of the graph.

For this type of graphs, it should be divided into two parts at the narrowest section.

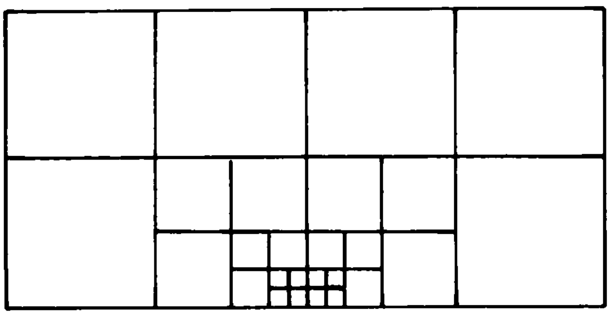
And the proposed method is indedendently applied to them, though the direction of the diameter of original graph must be held in the divided area.

These treatments can lead to the saving of work to obtain the critical cycles.

The proposed method can be applied to framed structures as shown in example–1 and –3.



(a) Regularly Subdivided Plate
(Number of Nodes = 45)



(b) Irregularly Subdivided Plate
(Number of Nodes = 42)

Fig. 7−1 Different Subdivisions of A Plate

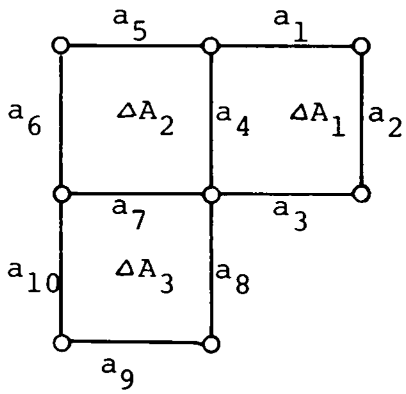
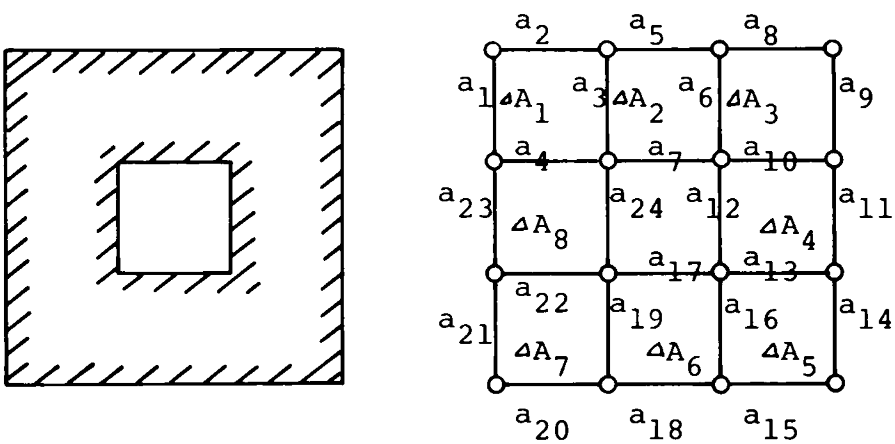


Fig. 7–2 A Plate with Three Elements



(a) A Plate with A Hole (b) Subdivided Plate

Fig. 7–3 A Plate and its Subdivisions

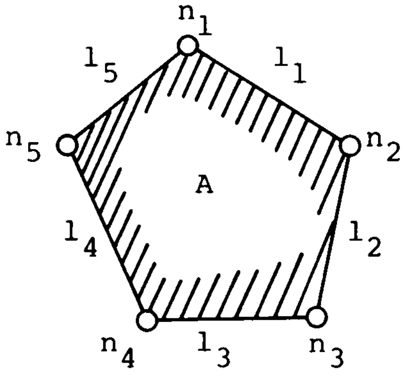
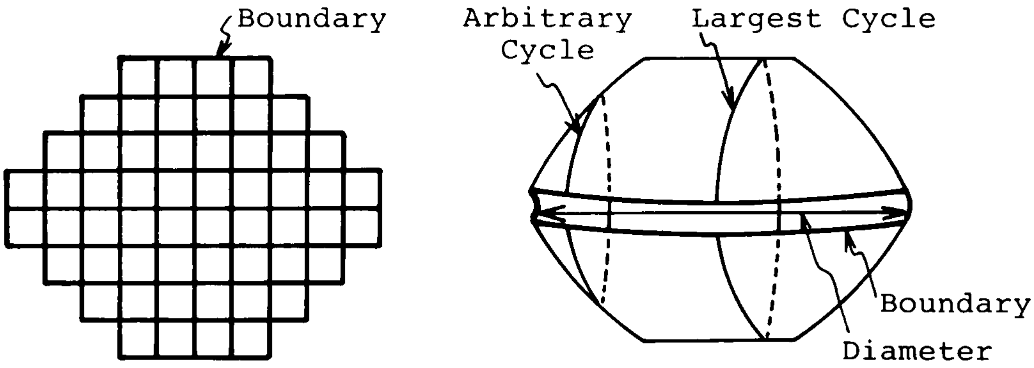


Fig. 7–4　A Surface and its Boundary



(a) Original System

(b) A Configuration in Three-Dimensional Field

Fig. 7-5 Arbitrary Configuration of A Plate in Three-Dimensional Filing Field (I)

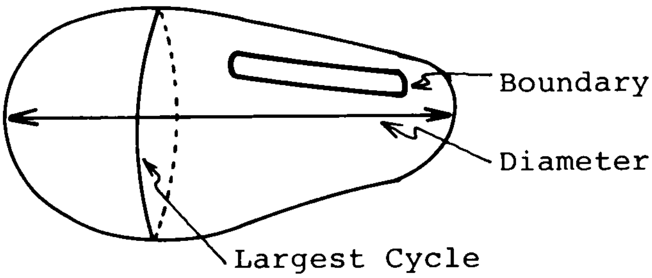


Fig. 7-6 A Configuration of A Plate in Three-Dimensional Filing Field (II)

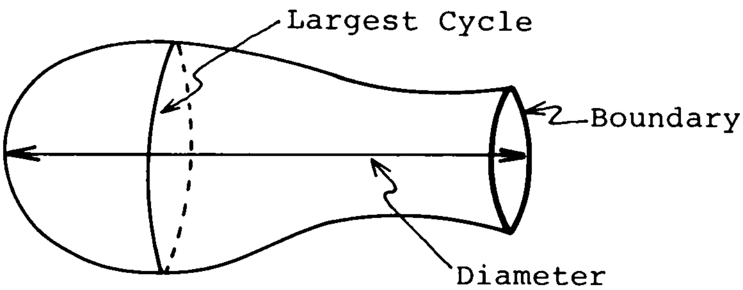


Fig. 7-7 A Configuration of A Plate in Three-Dimensional Filing Field (III)

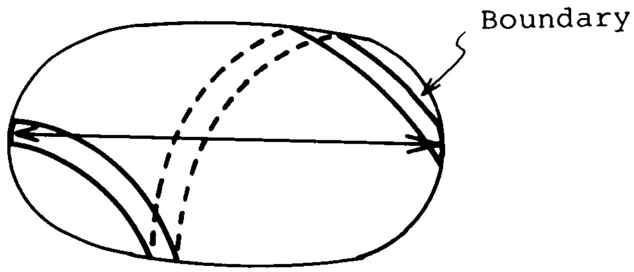


Fig. 7–8 Configuration of Boundary by Twisting Procedure for A Configuration in Three-Dimensional Filing Field

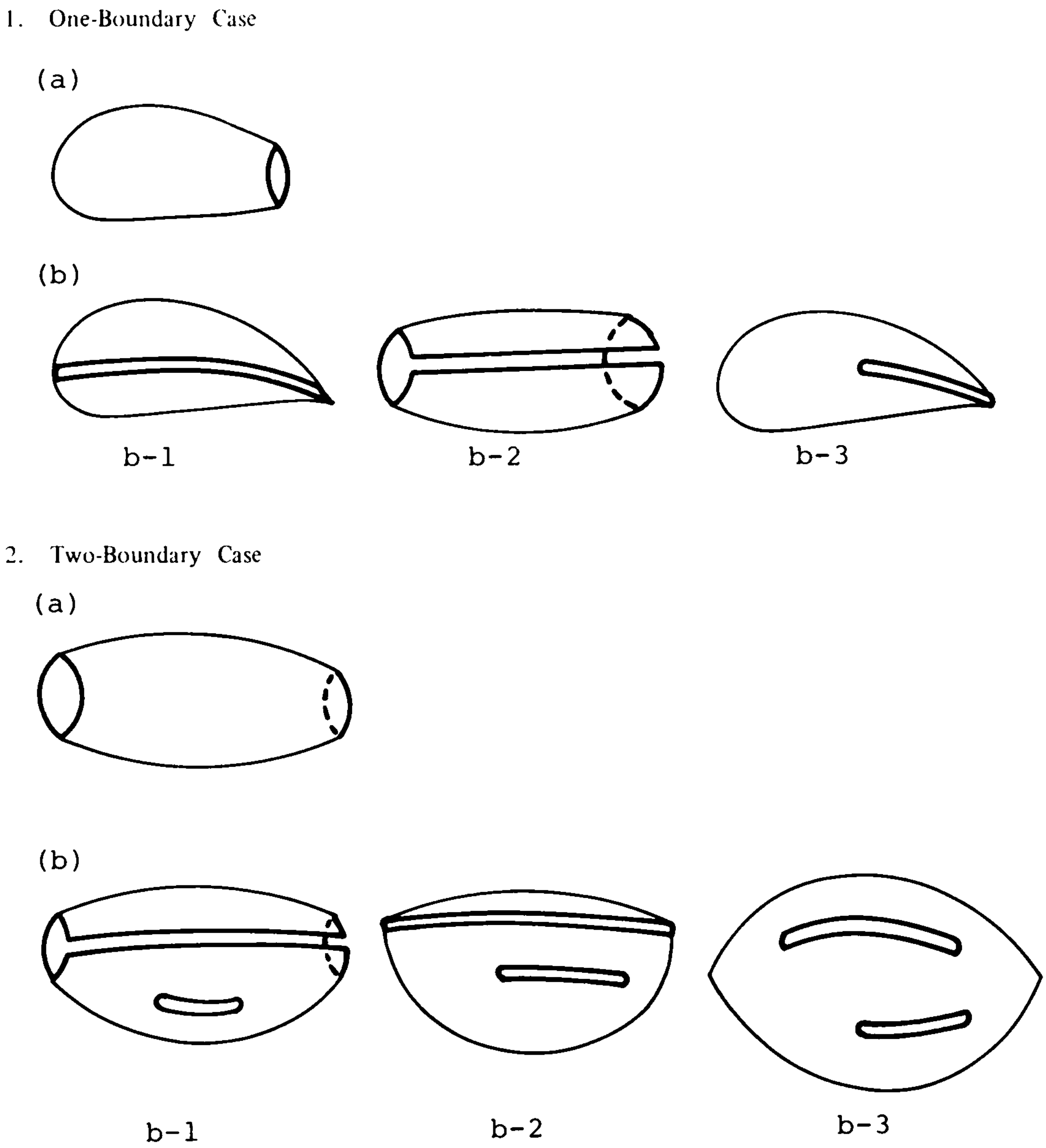
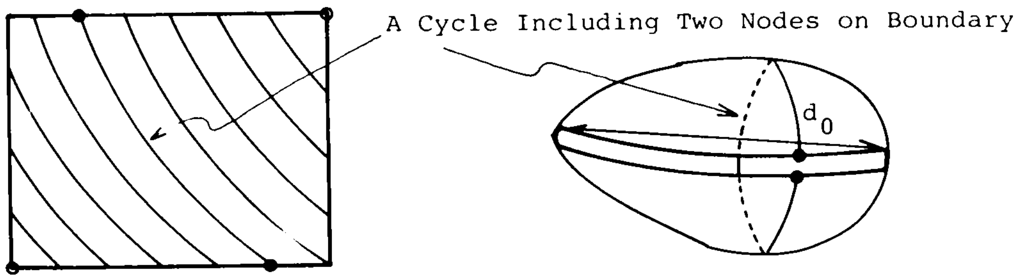
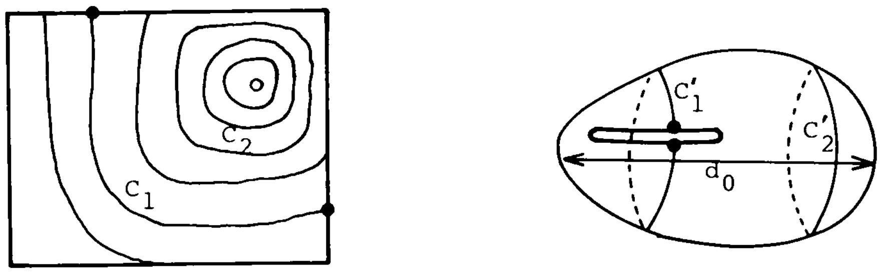


Fig. 7-9 Some Typical Configurations of A Plate-Like Structure in Three-Dimensional Filing Field



(a) Original Structure (b) Image in Filing Field

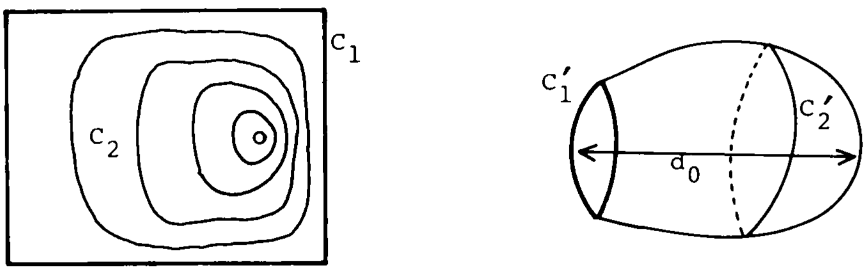
Fig. 7-10 Relation between Original and Filed Configuration (I)



(a) Original Structure (b) Image in Filing Field

C_1 and C_2 correspond to C_1' and C_2', respectively.

Fig. 7-11 Relation between Original and Filed Configuration (II)



(a) Original Structure (b) Image in Filing Field

C_1 and C_2 correspond to C_1', and C_2', respectively.

Fig. 7-12 Relation between Original and Filed Configuration (III)

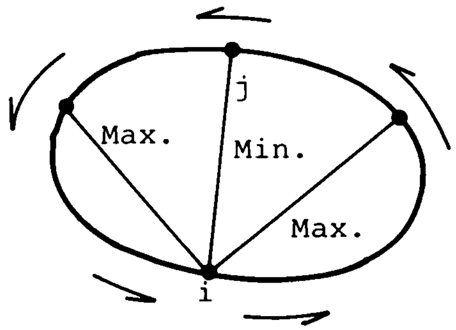


Fig. 7 13 Change of Value of The Shortest Path Connecting Two Nodes on Boundary

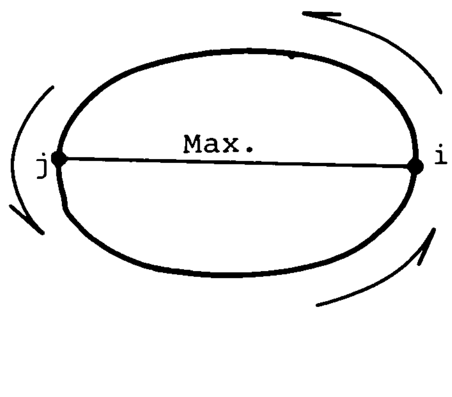
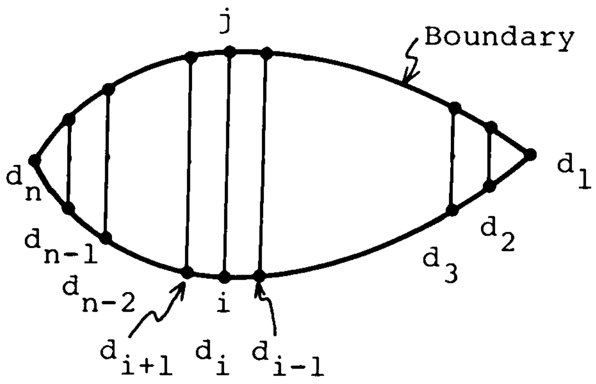
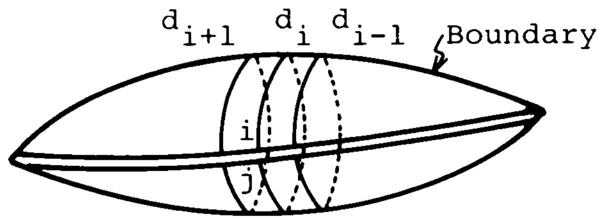


Fig. 7–14 Special Case of Shortest Path with Only One Maximum Value (The path corresponds to the diameter.)

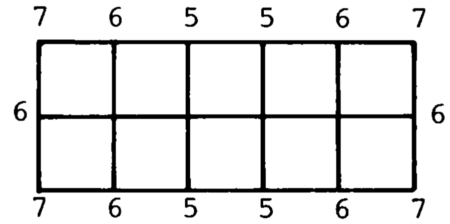


(a) Cycles in Original Configuration

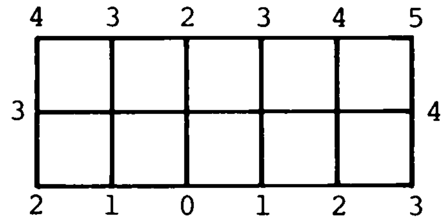


(b) Cycles in Three-Dimensional Filing Field

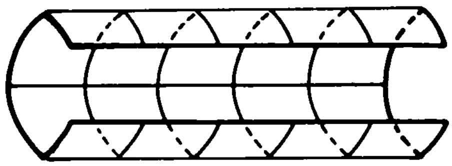
Fig. 7–15 Cycles in Original System and in Filing Field



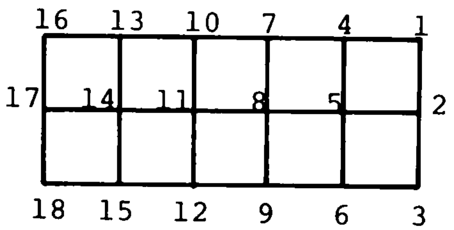
(a) Shortest Pathes Connecting Two Nodes on Boundary



(b) Distance from A Node on Boundary

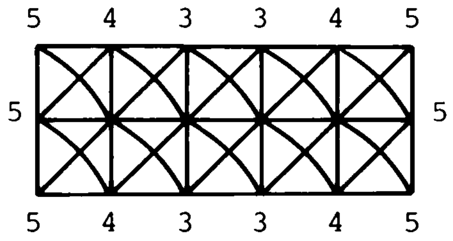


(c) Configuration in Filing Field

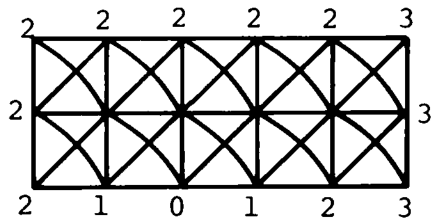


(d) Nodal Labeling with Minimum H. B. W. (H. B. W. = 4)

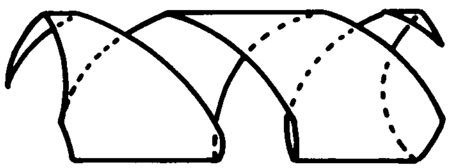
Fig. 7-16 Application of Bandwidth Reduction Method to Framed Structure



(a) Shortest Pathes Connecting Two Nodes on Boundary

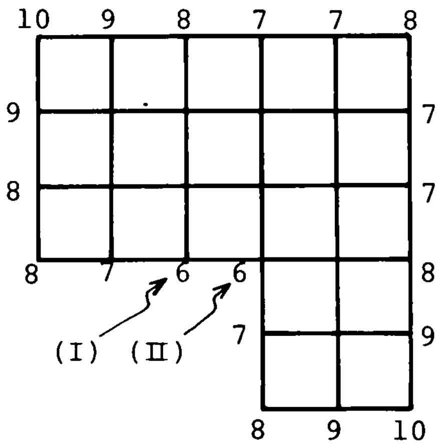


(b) Distance from A Node on Boundary

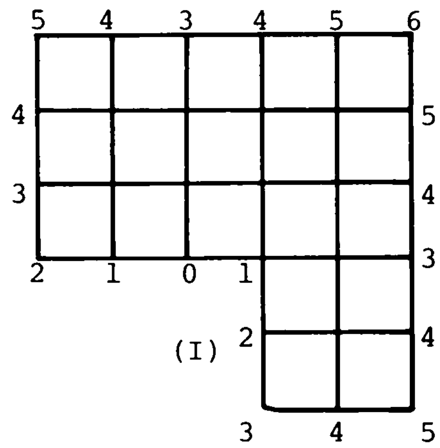


(c) Configuration in Filing Field (H. B. W. = 5)

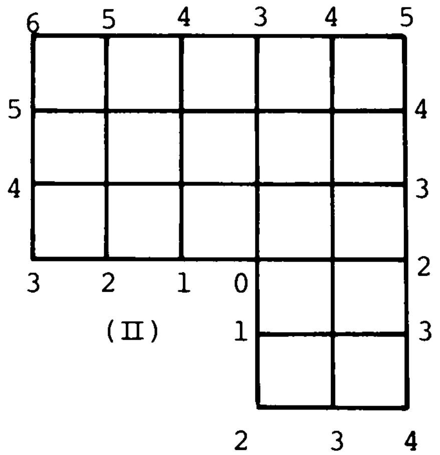
Fig. 7–17 Application of Bandwidth Reduction Method to Plate Structure with 10 Elements



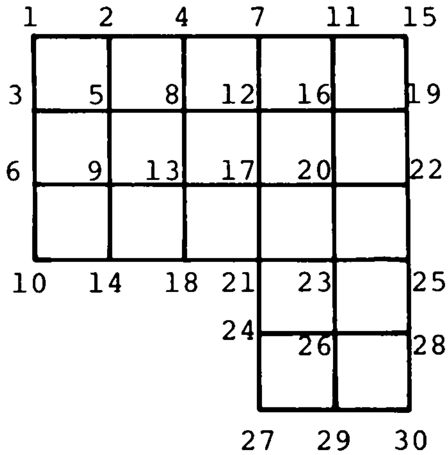
(a) Shortest Pathes Connecting Two Nodes on Boundary



(b) Shortest Pathes from (I)

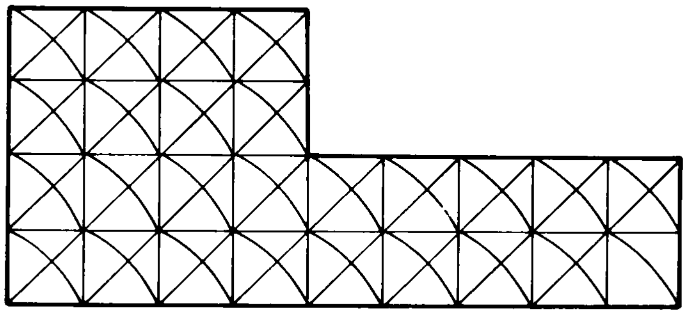


(c) Shortest Pathes from (II)

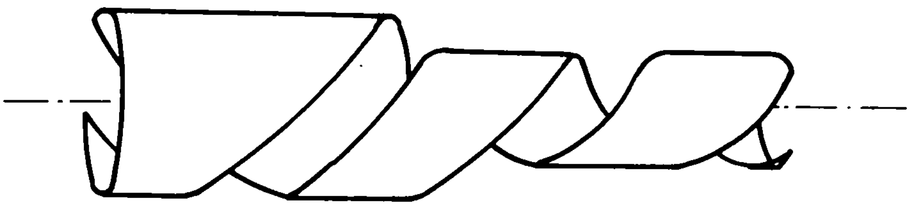


(d) Node-Labeling with Minimum Bandwidth (H. B. W. = 5)

Fig. 7-18 Bandwidth Reduction Procedure for Framed Structure with 30 Nodes

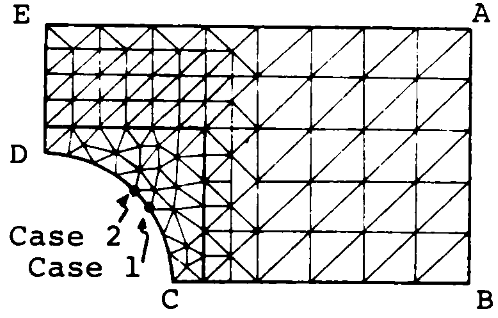


(a) Original Plate Structure with 26 Elements and 40 Joints

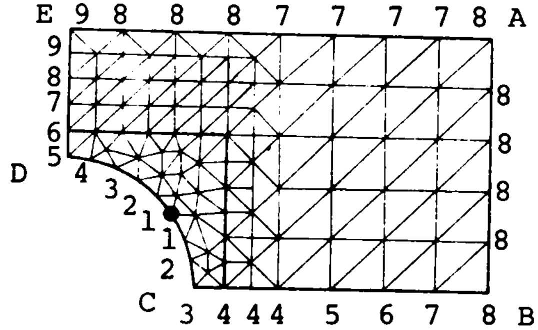


(b) Filed Configuration in Filing Field (H. B. W. = 7)

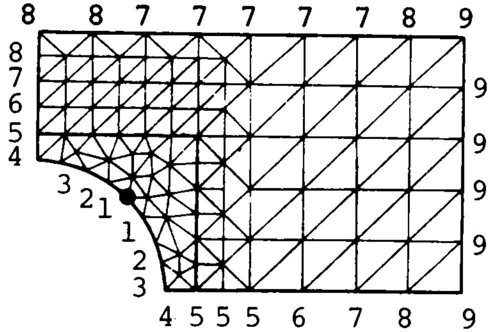
Fig. 7–19 Original Plate Structure and its Configuration with Minimum Radius in Filing Field



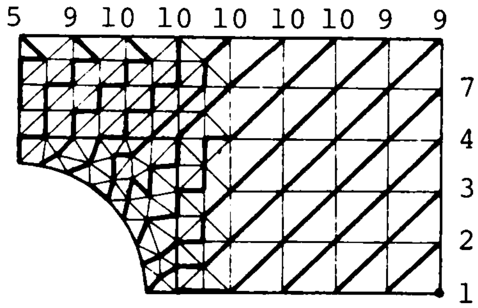
(a) Shortest Pathes Connecting Two Nodes on Boundary



(b) Shortest Pathes for Case 1

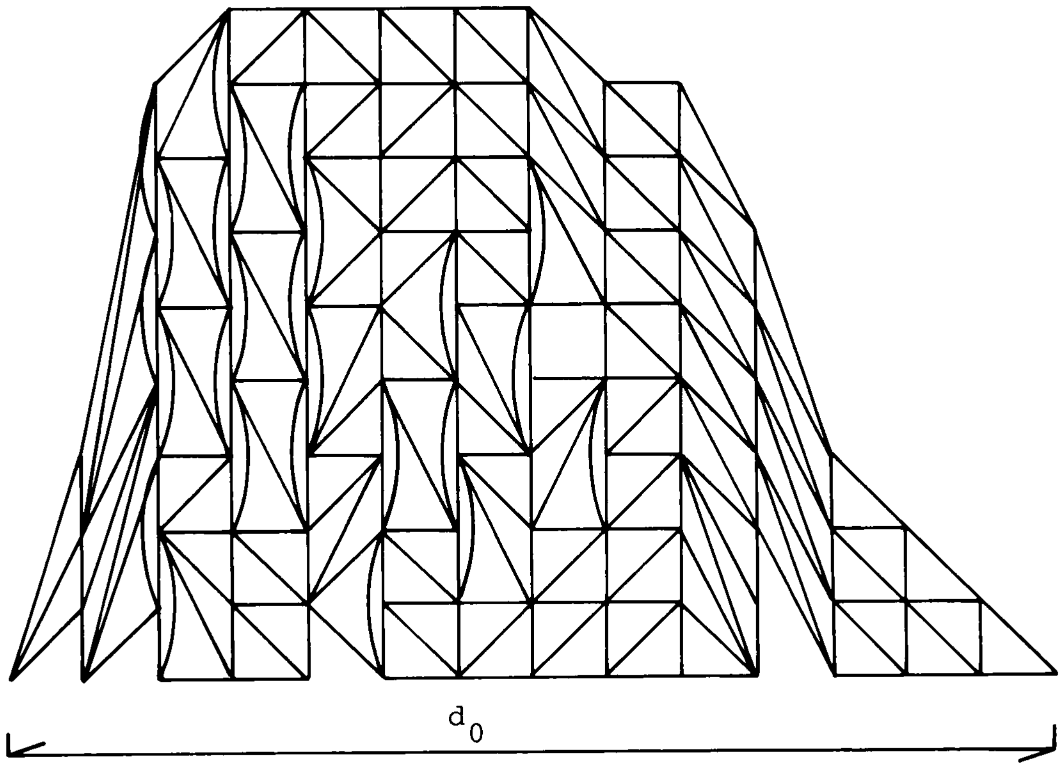


(c) Shortest Pathes for Case 2



(d) Actual Cycles in Original System (H. B. W. = 13)

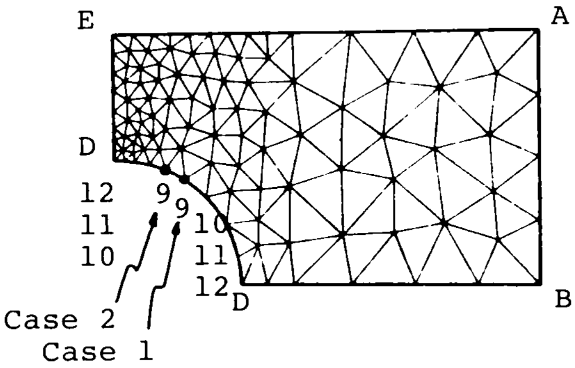
Fig. 7–20 Bandwidth Reduction Method Applied to A Plate with 99 Nodes



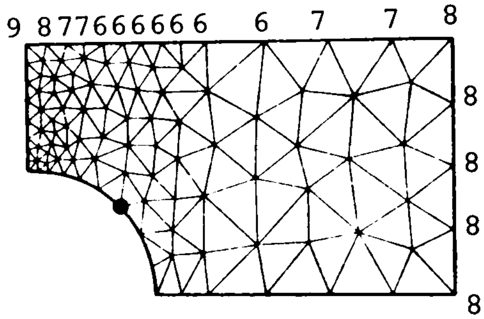
H.B.W. = 10 + 2 + 1

=13

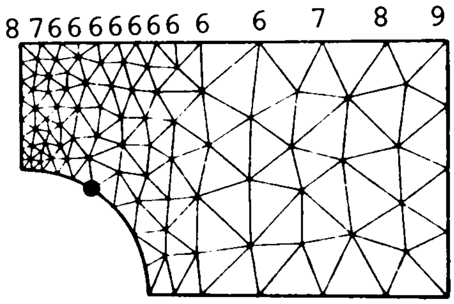
Fig. 7–20 (e) Filed Configuration in Two-Dimensional Field



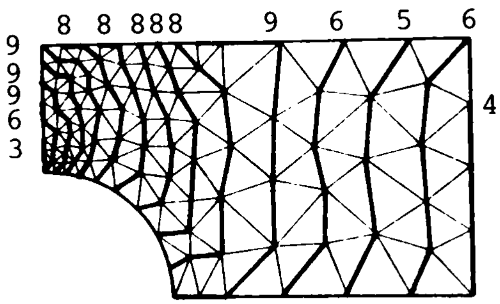
(a) Shortest Pathes Connecting Two Nodes on Boundary



(b) Shortest Pathes for Case 1



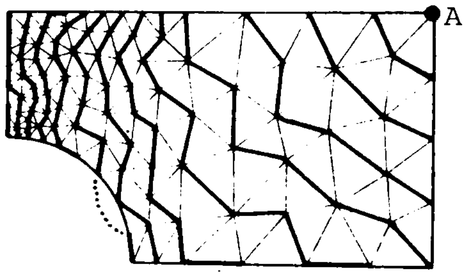
(c) Shortest Pathes for Case 2



(d) Actual Cycles in Original Graph

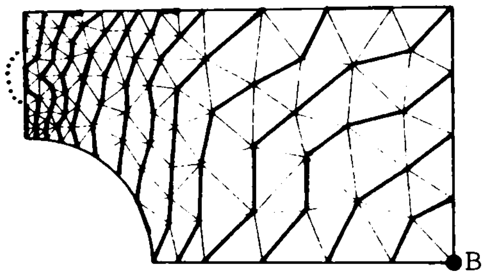
H. B. W. = 11

Fig. 7–21 Bandwidth Reduction Procedure Applied to A Plate with 106 Nodes



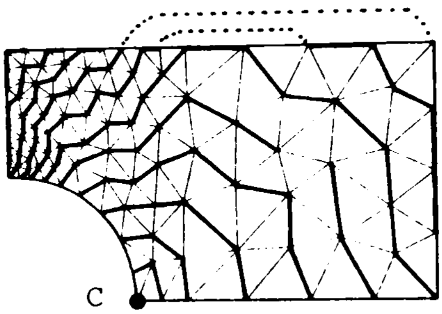
(e) Cycles Constructed from A by d = 1.

Max. Cycle = 11



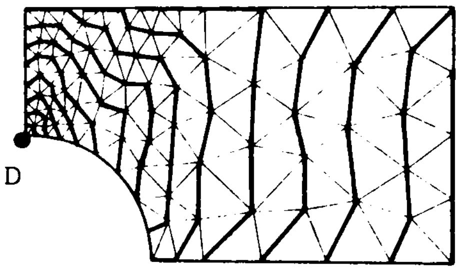
(f) Cycles Constructed from B by d = 1.

Max. Cycle = 12



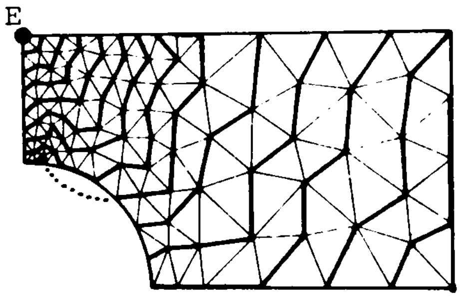
(g) Cycles Constructed from C by d = 1.

Max. Cycle = 17



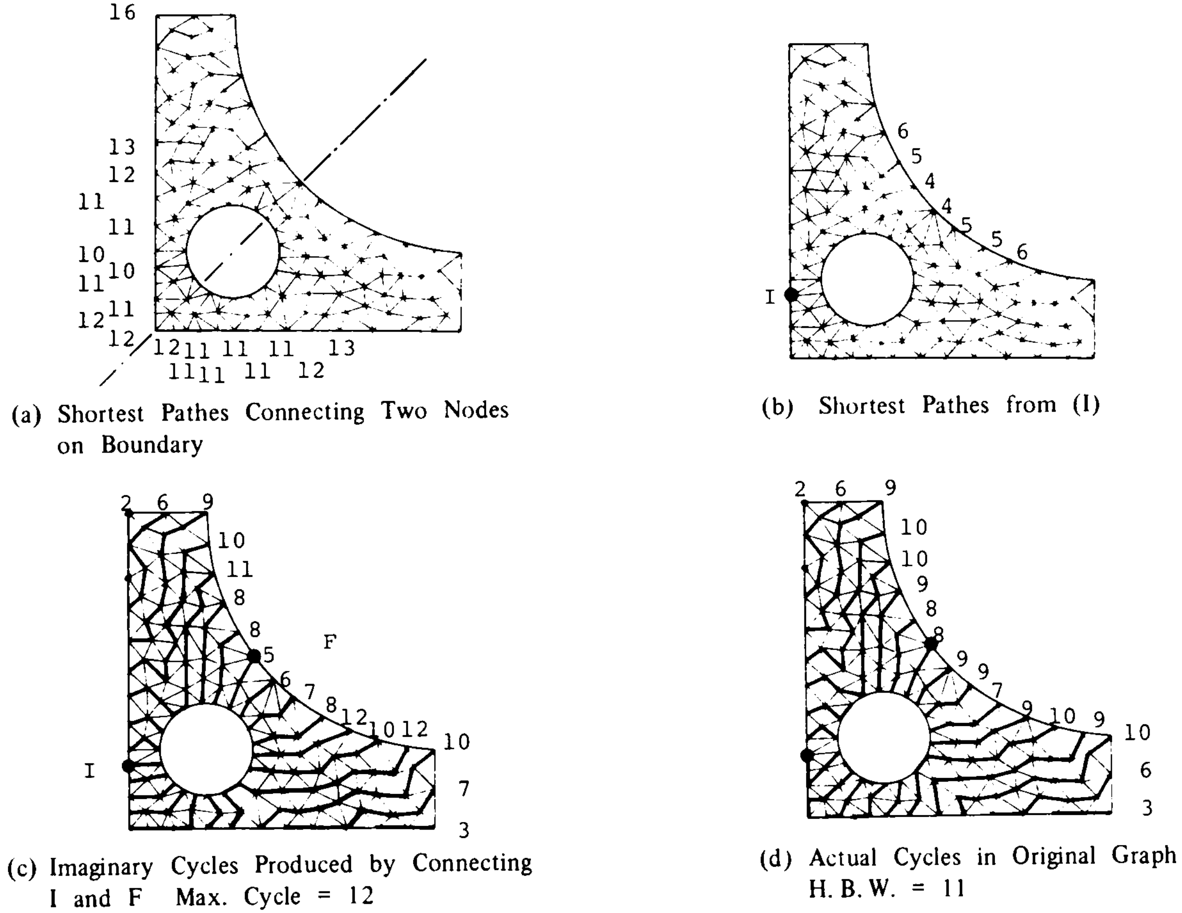
(h) Cycles Constructed from D by d = 1.

Max. Cycle = 14



(i) Cycles Constructed from E by d = 1.

Max. Cycle = 14



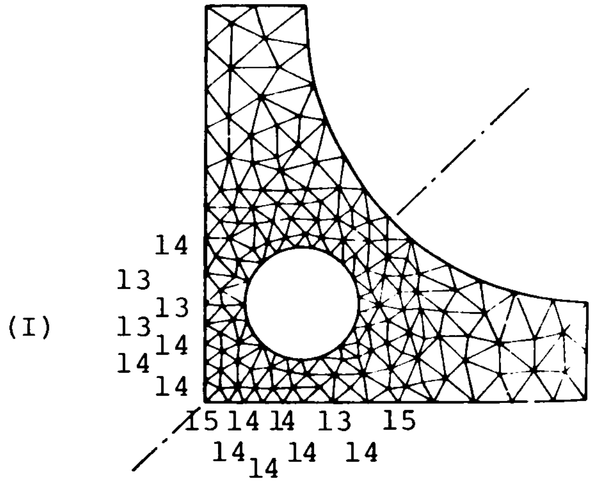
(a) Shortest Pathes Connecting Two Nodes on Boundary

(b) Shortest Pathes from (I)

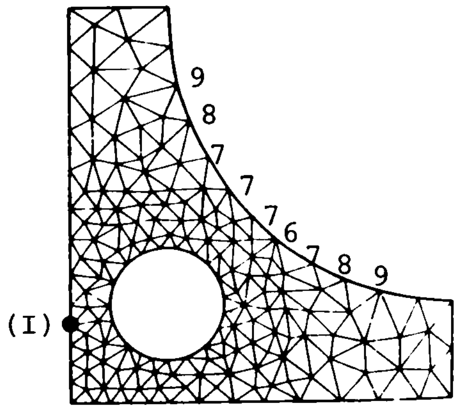
(c) Imaginary Cycles Produced by Connecting I and F Max. Cycle = 12

(d) Actual Cycles in Original Graph H. B. W. = 11

Fig. 7–22 Bandwidth Reduction Procedure Applied to A Plate with A Hole

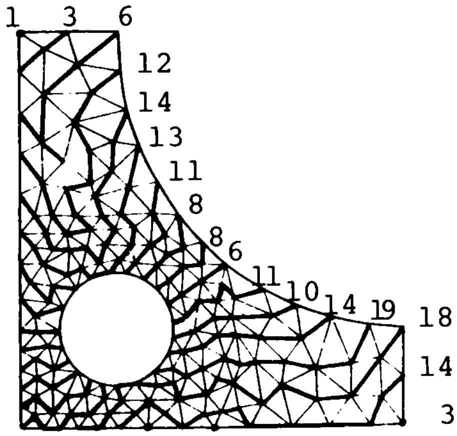


(a) Shortest Pathes Connecting Two Nodes on Boundary

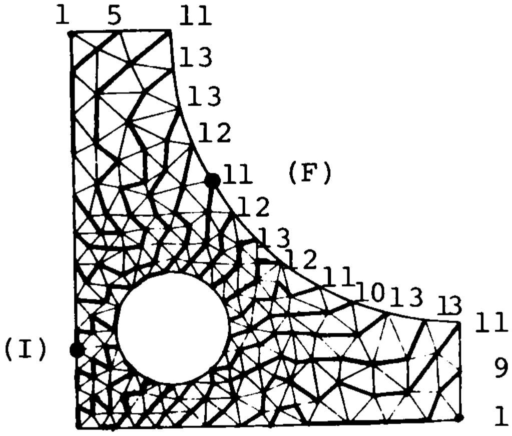


(b) Shortest Pathes from (I)

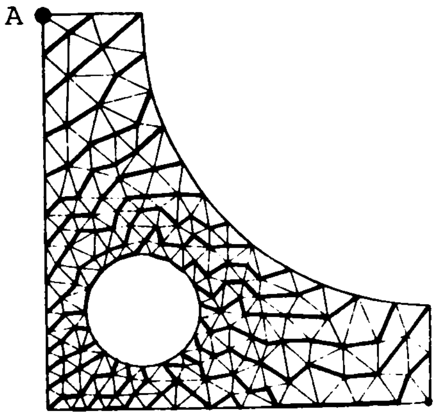
Fig. 7–23 Bandwidth Reduction Procedure Applied to A Plate with A Hole



(c) Cycles Disregarding Symmetricity of The Configuration (Max. Cycle = 13)



(d) Actual Cycles in Original Graph (Max. Cycle = 13)



(e) Cycles Constructed from A by d = 1 (Max. Cycle = 18)

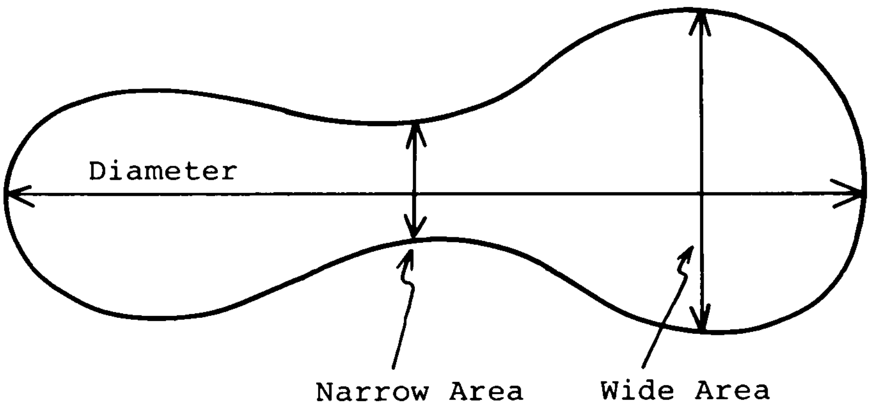


Fig. 7–24 An Example of A Graph which is unfavourable for The Proposed Method

CHAPTER 8

GRAPHICAL REPRESENTATION OF PROFILE OF STIFFNESS MATRIX

8–1. Introduction

In previous chapters in this thesis the author shows that the width of a graph corresponding to a structural system is just related to the bandwidth of stiffness matrix of the system.

If a graph with uniform nodal density has the convex boundary configuration, for example a circular configuration, the bandwidth is decided only by the width of the graph, i.e. its diameter. When the nodes in the graph are labeled in accordance with the mapped graph in the filing field, there exist a lot of zero elements within the bandwidth which cann't be reduced any more.

On the other hand we know another efficient inversion procedure of a matrix which treats only these elements which are located between the first non-zero element and the diagonal for every row matrix. This procedure can show its efficiency when the number of elements for input data is decreased to its minimum.

The number of elements enclosed in the area is called "Profile" of the matrix and the minimization of the value is called "Profile Minimization". Profile is, thus, given by following equation.

$$p = \sum_{i=1}^{n} (i - \alpha_i) \tag{8-1}$$

, where P ; profile of the matrix

i ; the i-th row matrix

α_i ; the column number where the first non-zero exists in the i-th row matrix

n ; row number

The value of P is changeable in accordance with the nodal labeling in the system.

Studies for the minimization procedure of profile were scarecely done,[40] but S. Yokota and H. Imai found interesting results for tree graph.* They showed that the minimum value of the profile for any tree graph can be obtained when the graph is given. Their results depend wholly on the distinguished characteristic of tree graph. That is, only one path exists between every two nodes in a tree graph, though there exist more than two pathes for any mesh graph. Thus, their results for tree cann't be directly applied to general mesh graphs.

* : Their results will be appeared as their Master Thesis and Bachelor Thesis, respectively.

In Chapter 4 the author proposed the filing fields in which any graph could be arbitrarily drawn. That is, in the field we can present a graph with arbitrary nodal ordering. In the sense, a graph can be expressed in the field as the profile of the graph is minimized.

In this chapter, the author investigates the relation between characteristics of graphs and their minimum values of profile, and he trys to express them by the aid of the filing fields. Moreover, he makes clear the relations between the configurations of graphs and their minimum half bandwidthes and also their minimum profiles.

8–2. Graph and Its Minimum Profile

A connected graph with "n" nodes and "m" lines is expressed by G(n, m). Before obtaining this graph, the preparatory work given in Section 5–2 must be, of course, operated for the given system.

The K-matrix which corresponds to the structural stiffness matrix of the system includes $(n + 2m)$ non-zero elements and the others, i.e. $(n^2 - n - 2m)$, are all zeros. Profile of the graph is defined by eq. 8–1.

Profile is thought as a kind of area in K-matrix, and among the value, P, there are only "m" non-zero elements and the others are all zeros which are enclosed by α_i and the principal diagonal line. Thus, the profile minimization is how to decrease the number of zeros in the area. The author calls the number of zeros in the area "loss" and he denotes it by "L_0".

Here, the author presents some typical graphs whose minimum profiles are easily obtained by our experiences and he gives some considerations for the profile minimization of tree graphs.

If a given graph is a complete graph, the K-matrix does not include any zero element and it suggests that the nodal labeling is arbitrary. And the profile of the complete graph, $G_C(n)$, is given by

$$P = n(n-1)/2 \qquad (8\text{–}3)$$

That is, the graph has no loss for any nodal labeling.

If a line is removed from $G_c(n)$, the diameter of the graph increases by one. This operation produces one zero element in the lower triangular matrix of K, and it should be placed at the first column matrix in K. Otherwise, the K-matrix includes one loss, i.e. $L_0 = 1$. Thus, at the stage of nodal labeling, one end of the removed line should be labeled by "1", and the others may be arbitrarily ordered.

Fig. 3–5 presents two typical tree graphs. In Fig. 3–5–a the centre is labeled "5" and the loss in this case is equal to 6, i.e. $L_0 = 6$. These zeros are produced by the labeling of the centre. If the node is labeled "8" or "9", the loss in K disappears. On the contrary, if it is labeled "1", L_0 increases to its maximum. Thus, we can conclude that the labeling of centres must be carefully done and it influences the value of profile.

Furthermore, the nodes which are located by d = 1 from a centre produce no loss, and their labeling are arbitrary.

Optimum states of the graph in the two-dimensional filing field are shown in Fig. 8–1–a and b. (c) shows that the centre is labeled "7", and the dotted line indicates the appearance of one loss in K-matrix. (d) presents the nodal labeling of K_I in Fig. 3–5–a, and 6 dotted lines appear in the figure and they correspond to the loss in K_I. Furthermore, we know that by the addition of the dotted lines all of the allowable lines between the second and third nodal columns can be picked up in the graph in the field. This fact indicates that the value of the loss is the number of residual allowable lines which are not given in original graph but are produced by the filing in the field.

Fig. 3–5–b shows a tree which is a nodal sequence. K_{II}-matrix in the figure has not any zero element in its profile and we know that the ordering is the best. The profile at the nodal labeling has evidently its minimum value. Thus, a nodal sequence in which a number of nodes are connected each other in series should be successively labeled from its one end to another. Optimum states of the graph are shown in Fig. 8–2–a, b, c and d. But, (e) and (f) don't present the optimum nodal ordering with minimum profile, but they include one and four losses, respectively. And the loss coincides with the dotted lines in (e) and (f).

Fig. 8–3 shows a general tree graph with only one centre. (b) shows the optimum filing state of the graph which includes least number of loss, i.e. $L_0 = 2$, but the state in (c) is not the best one and the loss increases by one, i.e. $L_0 = 3$. The increase of loss is caused by that nodes of a nodal sequence included in the nodal column are not successively arrayed. The loss is expressed by a dotted line between 8 and 9 in (c). But, the loss in a nodal column does not influence on the centre but it is restricted within the column. The loss at the centre is decided only by the number of nodes in the nodal column and the degree of the centre. Thus, at the centre we can give following equation for the loss.

$$L_0 = \sum_{i=1}^{\alpha-2} q_i - \alpha + 2 \qquad (8\text{–}4)$$

in which α ; degree of the centre

q_i ; the i-th nodal sequence at the centre

From this equation we know that the longest two nodal sequences should be selected among the sequences at the centre in order to reduce the loss at the centre to its minimum and they are placed in lateral direction in the filing field. This new nodal sequence which is composed by the selected two longest nodal sequences is called "trunk" of the graph. The selection of these two can be done by removing the centre-node of the graph and comparing the lengthes of α divided subgraphs. This procedure is possible only for tree graph, because all of the nodes in tree graph are cut-nodes whose removal induces the discretization of original graph as many subgraphs as the number of the degree of the node.

In the stage of labeling, nodes in the trunk are independently ordered from the nodes in the residual sequences.

In these considerations it becomes obvious that the loss of a tree graph appears only at centres and also that the profile minimization of a tree graph is the profile minimization at every centre.

Summarizing the configurations in the filing field, the loss appears between every neighbouring two nodal columns or between nodes in a nodal column.

General tree graph is a combination of the typical tree graphs in Fig. 3-5. Thus, in the labeling of a tree graph we should pay attention to the centres which produce the loss, and other nodes have no influence on the value of profile. The treatment of a general tree graph is given in Section 8–3.

8–3. Profile Minimization for Tree System

In this section the author shows how to minimize the profile of tree graph corresponding tree structural system.

A tree graph is denoted by G(n, n – 1) except the lines connecting to the datumn node. Suppose that the graph includes α centres.

Followingly, the author proposes the procedure of the profile minimization for a general tree graph. It is obtained by the considerations in the last section.

[Step–1]. Find out the main trunk of the whole graph.

Every centre in the graph is removed and the subgraphs around the centre are compared each other. The largest two subgraphs among them are selected. Or, we may select more than two subgraphs if they correspond to the largest two.

This treatment is repeated for all centres. After this operation, we begin to find out the trunk of the graph. Select any centre and examine whether the biggest two nodal sequences of the centre coincide with the biggest two nodal sequences from the neighbouring centres or whether they can be reached to the end-node (i.e. deg. = 1).

If the nodal sequence doesn't coincide with the one from the neighbouring centre, it can be concluded that the selected centre is not included in the trunk of the graph. Thus, we select another centre and repeat the above procedure till both ends of the nodal sequence connecting every two sequences at centre can reach at the end-nodes of the graph. The trunk which is found in this step is called "the main trunk" of the graph.

[Step–2]. Find out the sub-trunks of every subdivided graphs.

After Step–1, the nodal sequences which are selected as main trunk are removed from the original graph, and we obtain a number of subgraphs. They are treated independently in this step.

For every subgraph the procedure of finding out the trunk in Step 1 is repeated, and we obtain the subtrunk for every subgraph. These subtrunks are called "the first trunk".

After the selection of the first trunk, the nodal sequences which are included in the first trunk are removed and we obtain, again, more smaller subgraphs. These residuals are treated for the selection of "the second trunk".

This operation is repeated till the residuals include only one centre as shown in Fig. 5–3, because it can be treated as explained in the last section.

[Step -3]. Nodal labeling procedure of the graph.

Nodal-labeling is begun from one end of the main trunk and the nodes on the trunk are labeled successively to the node which connects to the first centre on the trunk. And the next numerical number is given to an end-node on the first trunk. Again, the labeling is successively repeated to the nodes on the trunk till we reach at the node which connects to the centre of the trunk. Labeling continues to the nodes on the second trunk. This procedure is repeated up to the highest trunk whose labeling is already shown in Fig. 8–3. After the labeling of the highest trunk, the labeling procedure goes downward till it reaches at the centre of the main trunk. By this labeling method, the labeling of centre at each trunk is done at the latest among all the nodes which are included in the subgraphs of the trunk.

By this labeling procedure, the loss appears at the centres and the value of the loss at every centre is always equal to the total number of nodes which are included in higher trunks. Thus, we obtain the equation for the total loss of the graph.

$$L_0 = \sum_{i=1}^{\alpha} (L_0^i) \tag{8–5}$$

, in which L_0^i ; the loss at the i-th centre and it is equal to the number of nodes which are included in higher trunks

α ; the number of centres

At this stage the author gives an example of the application of the proposed method to a tree graph and he shows how to obtain the minimum profile.

[Example]

The author applys the proposed procedure to a tree graph which is given in Fig. 5–29 in Chapter 5.

The result is shown in Fig. 8–4. It shows the given tree graph with 24 nodes and 23 lines. This graph contains 6 centres. In the graph, thick and thin lines indicate the main and the first trunks, respectively. Dotted lines are the residuals after the selection of the trunks.

In this case the loss appears only at the centres being labeled "19", and the value is equal to 6, i.e. $L_0 = 6$. The other centres don't produce any loss. There are 23 lines in the graph, and we obtain the result of minimum profile as following.

$$P = 23 + 6 = 29$$

Comparing this value with the results by use of other algorithms in Fig. 5–29, the efficiency of the proposed method for the profile minimization is obvious.

8–4. Graphical Approach to the Profile Reduction Method for Mesh System

Mesh graph has quite different characteristics comparing with those of tree graph and the proposed procedure for the profile minimization in the last section cann't be applied to mesh graph. The method depends wholly on the distinguished topological property of tree which can be divided into independent subgraphs by the removal of any node in the graph. That is, tree is the one which is composed by the least number of lines and whose nodes are cutnodes. This fact is not right for any general mesh graph. It cann't be subdivided by the removal of a node.

In this section the author trys to appreciate the profile (or the loss) of any mesh graph by the aid of the filing field which is introduced in Chapter 4, and he investigates how to reduce the loss from the configuration in the field.

Followingly, the author investigates the factors which influence the profile reduction by use of typical mesh graphs.

[Example–1]. One-mesh graph with arbitrary number of nodes.

This example is given as the first example in Chapter 3 in this thesis. The graph with "n" nodes is denoted by G(n, n).

If n = 3, the graph is a complete one and it includes no loss for any nodal labeling. If n = 4, we obtain two kinds of nodal labeling and both of them indicate that only one loss appears. This procedure is applied for G(n, n) and we know that the loss at optimum nodal labeling is given by following equation.

$$L_0 = n - 3 \tag{8-6}$$

Typical filing states of the mesh are presented in Fig. 8–5. The graph contains 9 nodes and we obtain the value of loss by eq. 8–6.

$$L_0 = 6$$

Three filing states in Fig. 8–5 show the optimum states and the number of the dotted lines in each filing state is equal to 6, and we know that all of them are actually in the optimum. The dotted lines are given in accordance with the restriction of allowable directions for every node in the configurations. From (b) and (c) it is obtained that every node on the nodal sequence filed in a nodal column has no loss for itself, because it is connected to two nodes and they are labeled successively. But the nodes on the bottom nodal row is connected to the node on the right-side nodal column by a lateral line, and the node has the loss which is equal to the height of the nodal column. From this we can conclude that the loss appears when there exists a line which connect neighbouring two nodal columns.

[Example–2]. Two-mesh graph.

An example is given in Fig. 8–6. (a) and (b) show the optimum filing states of the graph with the minimum loss, but (c) shows that L_0 = 9. The increase of the loss is caused by the existence of the lateral line at the third nodal row. The lateral line added to intermediate nodes of neighbouring two nodal columns produces a lot of loss which are nearly equal to the number of nodes in the nodal column, though the node has no loss before the addition of the lateral line.

From this example we can conclude that the appearance of lines with lateral direction in the filing field should be avoided as possible as the graph allows. Lateral line should be placed between low nodal columns. That is, a given graph should be filed in the filing field as long as possible in the lateral direction.

By using this conclusion next mesh graph can be easily filed in the filing field as it has the least number of loss.

[Example–3]. General mesh graph.

Fig. 8–7 shows a simple mesh graph with convex boundary configuration. (a) presents the original mesh and, if the graph is filed in the filing field as it be, the loss is equal to 12. As the diameter of the graph is equal to 5, it can be stretched in lateral direction as shown in (b). For the configuration in (b), $L_0 = 11$. For (c) in Fig. 8–7, $L_0 = 18$.

From the results for the profile of this simple mesh graph, the rightness of the considerations in [Example–1]and [Example–2] was secured.

Comparing (a), (b) and (c), the minimum profile is obtained when the mesh graph is stretched along the lateral axis of the filing field as long as its diameter. This example teaches that the diameter may play the important role for the optimum filing procedure in order to reduce the loss.

For tree graph the selection of trunk decides the value of loss for the graph. By the proper selection of the main trunk for tree graph, the number of nodes which are arranged in nodal columns on the centres of the main trunk is minimized. The fact for tree is established for general mesh graph, too. For any node of a mesh graph in the lowest nodal row in the filing field, the height of the nodal column decides its loss. (See Fig. 8–7). Moreover, the fact is established for any nodal row in the filing field, because the loss of a node at (the i-th nodal row, the j-th nodal column) is decided by the number of nodes which locate below the i-th row in the (j – 1)-th nodal column and also by the number of nodes which locate above the node in the j-th nodal column. This is established when the node is connected to the node which locates at (the i-th nodal row, the (j – 1) th nodal column). In general, the first nodal column in the filing field does not produce any loss, because the nodes in the column have no neighbouring nodal column. For a node in a nodal column except the initial column the appearance of the loss is just shown in Fig. 8–8. From this figure we can give the value of the loss at the node.

$$L_0 = \alpha + \beta - \gamma \tag{8–7}$$

, in which γ is the number of lines which connect the (i, j) node to those nodes whose numerical ordering are smaller than that of the node.

[Example 4]. Mesh graph with concave boundary configuration.

The mesh graphs given in Chapter 6 correspond to this. For the bandwidth reduction they are modified into tree system. That is, their characteristic is similar to that of tree graph.

For the profile minimization they should be treated just like tree graphs. That is, the kind of mesh graphs is subdivided into main mesh trunk and secondary mesh branches. For the secondary mesh branches, they are independently treated as to minimize their loss. And for the main mesh trunk its loss is the summation of the loss of itself and the loss from the secondary mesh branches.

Above considerations of the profile reduction procedure for mesh graph with irregular boundary configuration were obtained from the results in Example 3 which suggests that the graph should be filed within fewer nodal rows in the filing field.

Summarizing the considerations which were done for the above examples yields to following items which are important factors for the profile reduction of mesh graphs.

1. As the concept of trunk is important for profile of tree graph, the longitudinal direction of given mesh graph plays the most important role for the reduction of profile for any mesh. The diameter of the graph may take the place. In the filing field the diameter (or the temporary diameter) of any mesh is placed along the lateral axis.
2. Any mesh should be filed as the least number of lateral lines appears in the filing field. At the same time the number of nodes in nodal columns where lateral lines appear should be decreased as the graph allows.
3. If the given mesh graph has a convex boundary configuration, the degree of freedom of each node in the filing field is small. Thus, the procedure of obtaining minimum cycles for the bandwidth reduction method in Chapter 7 can be applied for the profile reduction.
4. If the system has a concave boundary configuration, the profile reduction procedure for tree graph can be similarily applied. That is, the loss of the secondary mesh branches except the main mesh trunk should be independently minimized.

8–5. Conclusions.

In these investigations for the profile reduction of stiffness matrix, the author showed that the filing fields defined in Chapter 4 are effective and valid for the purpose of the profile reduction of tree and also mesh graph.

As the mesh systems with concave boundary configuration can be easily teated as trees in the bandwidth reduction problem, the kind of mesh systems may be treated as tree systems for the profile minimization problem, too.

Moreover, it was shown that the strategy for the profile reduction of mesh graph with convex boundary corresponds with the proposed method of finding minimum cycles for the bandwidth reduction in Chapter 7.

Comparing the bandwidth reduction and the profile reduction methods, the biggest difference appears when tree graph is treated. This is caused by the difference of the concepts of "width" for profile and bandwidth. For the former it indicates the total number of branch nodal sequences at every centre, and for the latter it means the average height in the filing field.

All of the examples given in this chapter have the uniform nodal density. If the analyst treats a graph with non-uniform nodal density, he has to, at first, modify the graph into the one with the uniform nodal density. And he has to investigate the configuration of the boundary. According to the boundary configuration the profile reduction method has to be selected as given in this section. This procedure is just same to the one for the bandwidth reduction method in previous chapters.

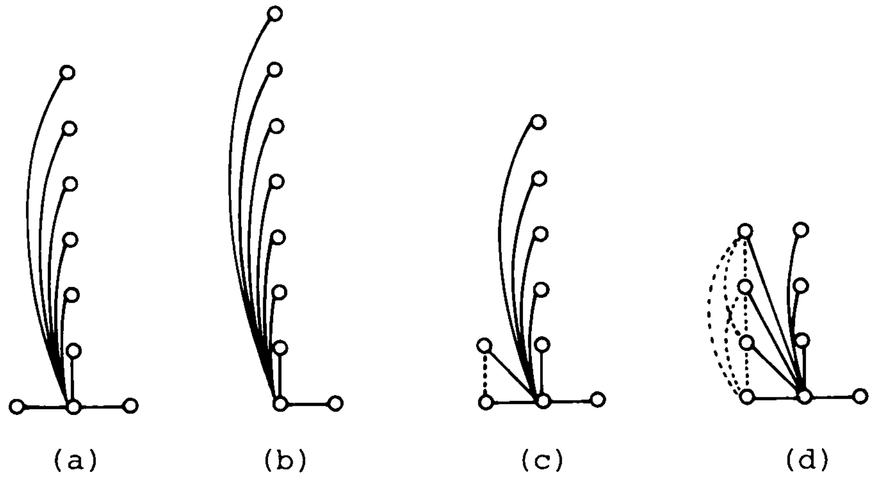


Fig. 8–1 Filing States of A Tree Graph with One Centre and The Appearance of Loss

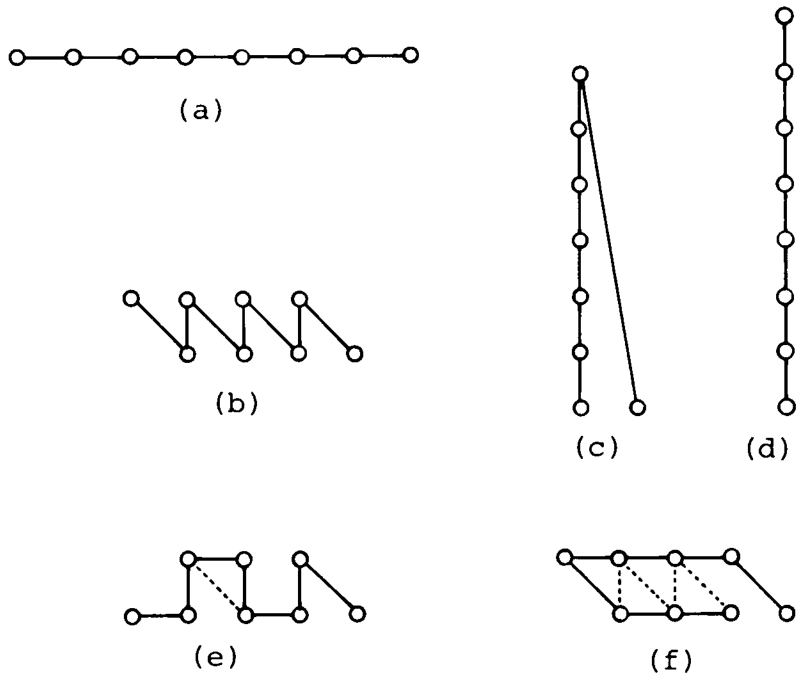
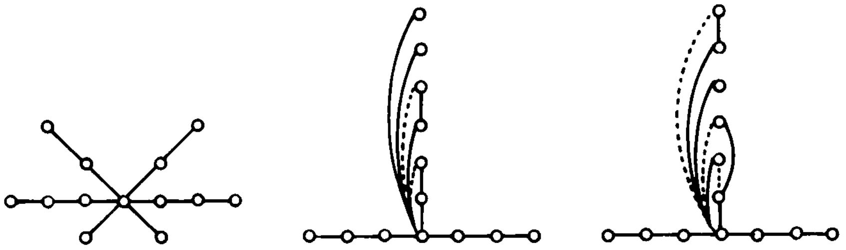


Fig. 8–2 Filing States of A Nodal Sequence and The Appearance of Loss



(a) Given Tree

(b) Optimum State with L_0 = 2

(c) Filing with L_0 = 3

Fig. 8–3 A General Tree Graph with One Centre

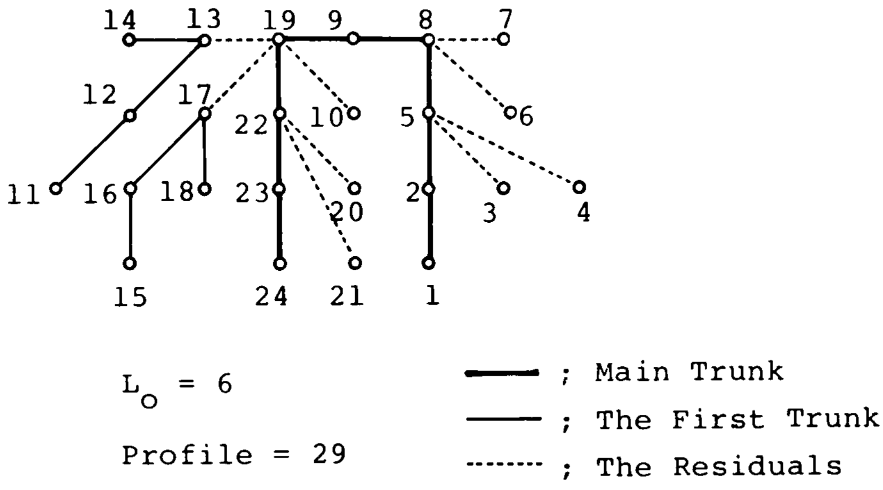
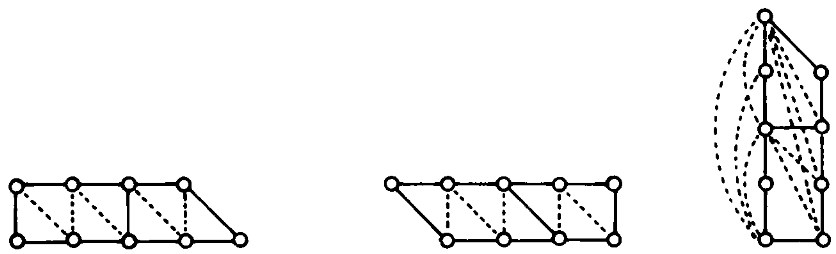


Fig. 8–4 Profile Minimization for Tree Graph with 24 Nodes and 23 Lines

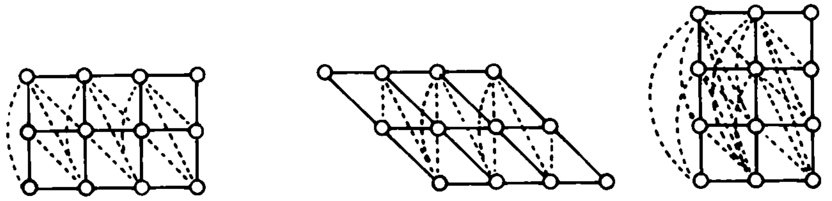
(a) (b) (c)

Fig. 8–5 Optimum Filing States of
A mesh Graph ; G(9, 9)
L_0 = 6, Minimum Profile = 15



(a) L_0 5 (b) L_0 = 5 (c) L_0 = 9

Fig. 8–6 Filing States of Two-Mesh Graph
Minimum Profile = 15



(a) $L_0 = 12$ (b) $L_0 = 11$ (c) $L_0 = 18$

Fig. 8–7 Mesh Graph in Filing Field
Minimum Profile = 28

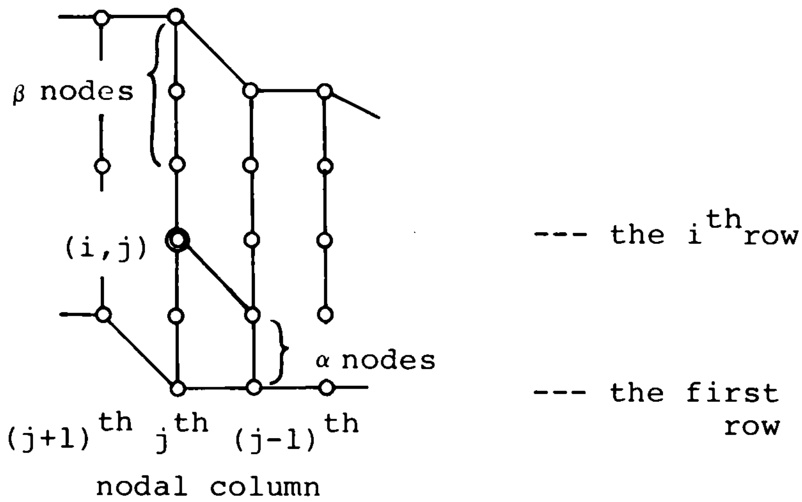


Fig. 8–8 Appearance of Loss between Neighbouring Nodal Columns in Filing Field

CHAPTER 9
CONCLUDING REMARKS

Through these investigations on the application of topology to the bandwidth reduction method for stiffness matrix of any structural system, following features are considered to be clarified.

1. The topology of a structure decides the minimum value of the bandwidth of stiffness matrix. Actually, the configuration of the connected graph which expresses the connectivity relationship between joints in a system gives the value, itself, though it is hidden in its complexity of lines, non-uniform distribution of nodes, and the complex shape of surrounding boundary of the system. Thus, the removal of those factors which induce the complexity for obtaining the minimum value yields to the obvious representation of the value by use of original graph. That is, the homomorphic mapping of original graph gives an image which shows the value obviously.
2. The definition of the filing field is the basis of the proposed bandwidth reduction methods. The bandwidth of a stiffness matrix of any structural system coincides with the maximum value along specified one axis of the filing field. Thus, the bandwidth reduction is replaced to how to reduce the value along the axis in the field.
3. The bandwidth reduction methods in Chapter 5, 6 and 7 are effective for tree structures, rather simple mesh systems, and general mesh structures, respectively. Especially, the methods in Chapter 5 and 6 are preferable for the structural systems with irregular boundary configurations, and the one in Chapter 7 is applicable when a system has non-uniform nodal distribution. Even if the original has simple boundary shape but it has non-uniform nodal distribution, the shape of the boundary becomes very complicated after the distribution of nodes is forced to be uniform. Thus, the procedure to make the nodal distribution uniform is, at first, applied to the system and after the procedure the methods in Chapter 5 and 6 should be applied.
4. The methods proposed in this thesis are mainly applicable for two-dimensional structures. Even if they have three-dimensional configuraitons, the methods can be applied, because the filing fields can admit any graphical configuration. But for simpler graphical representation, it may become necessary to define new filing field which fits for them. And a new bandwidth reduction method in the field is also required to be found. But it is guessed that the method is a kind of modification of the methods in this thesis. In the sense, the bandwidth of any structural stiffness matrix is supposed by the aid of the concepts which are introduced in the methods of this thesis.
5. Among actual structural systems, those which have obvious longitudinal direction are very easily examined their minimum bandwidthes, because they show apparently the direction of the diameter which is one of the most important factors to decide the

bandwidth. Bridge structures are good examples for this case.

If the direction of the diameter is vague in original system, it is very difficult to obtain the minimum value of its bandwidth. A plate with finite elements corresponds to this case.

If the shape of the boundary of a plate is, almost, convex, the method in Chapter 7 shows its efficiency, but if it shows a concave configuration of the boundry, the method decreases its efficiency although it is still superior comparing to methods proposed in past studies.

6. The definitions of filing fields can express the concept of "Profile" in structural stiffness matrix. The investigations in Chapter 8 show that the bandwidth reduction method in Chapter 7 can be valid for the profile reduction of mesh system with convex boundary configuration. In other word, there exists not so large difference between them. On the other hand, the results for tree system have large difference between the profile minimization and bandwidth reduction. But we can conclude that the difference comes from the difference of the concept of "width" of the graph.

7. The proposed methods require the graphical treatment and they need certain judgements by analysts with rich experience of nodal labeling. In the sense, they don't fit to digital computers. Thus, it is hoped that the studies in this field are proceeded in future, till the methods can be fitted to the characteristics of computers. Or, it can be said that the problem, itself, can not admit the features of digital computers. If it is right, the generality of the methods is impossible and, on the contrary, methods should be modified as to be applicable to only a kind of structural systems.

REFERENCES

1. Kron, G., Diakoptics ; The Piecewise Solution of Large-scale Systems, Macdonald, London, 1963
2. Kron, G., "A Set of Principles to Interconnect the Solutions of Physical Systems", Journal of Applied Physics, 24, 1953
3. Kron, G., "Tearing and Interconnecting as a Form of Transformation", Quarterly of Applied Mathematics, Vol. 13, No. 2, 1955 pp. 147 ~ 159
4. Branin, F.H., "The Relation between Kron's Method and the Classical Methods of Network Analysis", I.R.E. WESCON Convention Record, Part 2, 1959
5. Spillers, W.R., "On Diakoptics ; Tearing an Arbitrary System", Quarterly of Applied Mathematics, Vol. 23, 1965, pp. 188–190
6. Roth, J.P., "An Application of Algebraic Topology : Kron's Method of Trearing", Quarterly of Applied Mathematics, Vol. 17, 1959, pp. 1 ~ 24
7. Householder, A.S., "A Survey of Some Closed Methods for Inverting Matrices", Journal of Soc. Ind. and Appl. Math., Vol. 5, 1957, pp. 155–169
8. Fenves, S.J., and Branin, F.H., "Network-Topological Formulation of Structural Analysis", Journal of Structural Division, Proc. of ASCE, Vol. 89, ST 4, 1963 pp. 483 ~ 514
9. Spillers, W.R., "Application of Topology in Structural Analysis", Journal of Structural Division, Proc. of ASCE, Vol. 89, ST 4, 1963, pp. 301 ~ 314
10. DiMaggio, F.L. and Spillers, W.R., "Network Analysis of Structures", Journal of Eng. Mech. Division, Proc. of ASCE, Vol. 91, EM 3, 1965, pp. 169 ~ 188
11. Fenves, S.J., "Structural Analysis by Networks, Matrices and Computers", Journal of Structural Division, Proc. of ASCE, Vol. 92, ST 1, 1966, pp. 199 ~ 222
12. Kron, G., "Solving Highly Complex Elastic Structures in Easy Stage ", Journal of Applied Mechanics, 1955
13. Kron, G., "Tensorial Analysis and Equivalent Circuits of Elastic Structures", Journal of the Franklin Institute, Vol. 238, No. 6, 1944 pp. 400–453
14. Tamamura, S., Topological Considerations on Statical Analysis of Framed Structures, Master of Engineering Thesis, No. 326, Kyoto University, 1967 (in Japanese)
15. Konishi, I., Shiraishi, N. and Tamamura, S., "A Network-Topological Study on Statical Analysis of Rigid Framed Structures", Memoirs of the Faculty of Engineering, Kyoto University, Vol. 30, Part 2, April 1968, pp. 94 ~ 104
16. Konishi, I., Shiraishi, N., Tamamura, S. and Taniguchi, T., "A Network-Topological Study on Statical Analysis of Rigid Framed Structure", Memoirs of the Faculty of Engineering, Kyoto University, Vol. 31, Part 4, October 1969, pp. 441 ~ 455
17. Mauch, S.P. and Fenves, S.J., "Releases and Constraints in Structural Networks", Journal of Structural Division, Proc. of ASCE, Vol. 93, ST 5, 1967 pp. 401 ~ 418

18. Spillers, W.R., "Network Analogy for Linear Structures", Journal of Engineering Mechanics Division, Proc. of ASCE, Vol. 89, EM 4, 1963, pp. 21 ~ 29
19. Spillers, W. R., Automated Structural Analysis, Pergamon Press, 1972
20. Livesley, R. K., Matrix Method of Structural Analysis, Pergamon Press, 1964
21. Caro, A. G. and Fenves, S. J., "A Network-Topological Formulation of the Analysis and Design of Rigid Plastic Framed Structures", A Technical Report of a Research Program, University of Illinois, September 1968
22. Taniguchi, T., A Network-Topological Study on Statical Analysis of Framed Structures, Master of Engineering Thesis, Kyoto University, 1971
23. Konishi, I., Shiraishi, N., Taniguchi, T. and Furuta, H., "An Application of Network-Topological Concepts to the Design of Framed Structures", Memoirs of the Faculty of Engineering, Kyoto University, Vol. 36, Part 4, 1974
24. Niiseki, S. and Satake, M., "Some Applications of Topological Consideration and Weight-Matrix to Finite Element Analysis", Tokyo Seminar on Finite Element Analysis, 1973, pp. 61 ~ 73
25. Stewart, K. L. and Baty, J., "Dissection of Structures", Journal of Structural Division, Proc. of ASCE, ST 5, 1967, pp. 217 ~ 232
26. Thurnau, T. H., "Algorithm 195 ; BANDSOLVE", Communications of ACM, Vol. 6, No. 8, August 1963, pp. 441
27. Zienkiewicz, O. C., The Finite Element Method in Engineering Science, MacGraw Hill, 1971
28. Kikuchi, M., "Structural Analysis by Using Finite Element Method", The Bridge, September 1972, pp. 90 ~ 96 (in Japanese)
29. Spillers, W. R., "Techniques for Analysis of Large Structures", Journal of Structural Division, Proc. of ASCE, ST 11, 1968, pp. 2521 ~ 2534
30. Ingram, "On the Inversion of the Cauer-Routh Matrix", Quarterly of Applied Mathematics, Vol. 27, No. 2, 1969
31. Alway, G. G. and Martin, D. W., "An Algorithm for Reducing the Bandwidth of a Matrix of Symmetric Configuration", Computer Journal, Vol. 8, 1965, pp. 264 ~ 272
32. Rosen, R., "Matrix Bandwidth Minimization", Proc. of ACM National Conference, 1968, pp. 585 ~ 595
33. Akyuz, F. A. and Utku, S., "An Automatic Node-Relabeling Scheme for Bandwidth Minimization of Stiffness Matrices", AIAA Journal, Vol. 6, No. 4, 1968, pp. 728 ~ 730
34. Spillers, W. R., "Techniques for Analysis of Large Structures", Journal of Structural Division, Proc. of ASCE, ST 11, 1968, pp. 2521 ~ 2534
35. Spillers, W. R. and Hickerson, N., "Optimal Elimination for Sparse Symmetric Systems as a Graph Problem", Quarterly of Applied Mathematics, Vol. 25, 1968, pp. 425 ~ 432
36. Rose, D. J., "A Graph-theoretic Study of the Numerical Solution of Sparse Positive Definite Systems of Linear Equations", Graph Theory and Computing, Academic Press, 1972, pp. 183 ~ 217 (Editor ; R. C. Read)

37. Collins, R. J., "Bandwidth Reduction by Automatic Renumbering", International Journal For Numerical Methods In Engineering, Vol. 6, 1973, pp. 345 ~ 356
38. Kawamo, K., Wakasugi, T. and Oshizawa, K., "Efficient Solution of a Large Linear System for the Computer Analysis of Structure", Proc. of National Symposium of JSSC, 1971, pp. 1 ~ 9
39. Cuthill, E. and McKee, J., "Reducing the Bandwidth of Sparse Symmetric Matrices", Proc. of ACM National Conference, 1969, pp. 157 ~ 172
40. Cuthill, E., "Several Strategies for Reducing the Bandwidth of Matrices", The IBM Research Symposia Series ; Sparse Matrices and Their Applications, Plenum Press, 1972, pp. 157 ~ 166 (Editors ; D. J. Rose and R. A. Willoughby)
41. Grooms, H. R., "Algorithm for Matrix Bandwidth Reduction", Journal of Structural Division, Proc. of ASCE, ST 1, January, 1972, pp. 203 ~ 214
42. Tewarson, R. P., "Row-column Permutation of Sparse Matrices", Computer Journal, Vol. 10, 1967, pp. 300 ~ 305
43. Paton, K., "An Algorithm for the Blocks and Cutnodes of a Graph", Communications of the ACM, Vol. 14, No. 7, July 1971 pp. 468 ~ 475
44. Melosh, R. J. and Bamford, R. M., "Efficient Solution of Load-Deflection Equations", Journal of Structural Division, Proc. of ASCE, ST 4, April 1969, pp. 661 ~ 676
45. King, I. P., "An Automatic Reordering Scheme for Simultaneous Equations Derived From Network Systems", International Journal For Numerical Methods In Engineering, Vol. 2, 1970, pp. 523 ~ 533
46. Irons, B. M., "A Frontal Solution Program For Finite Element Analysis", International Journal For Numerical Methods In Engineering, Vol. 2, 1970, pp. 5 ~ 32
47. Harary, F., Graph Theory, Addison-Wesley Publishing Co., Kyoritsu Shuppan 1971 (in Japanese)
48. Bellman, R., Cooke, K. L. and Lockett, J. A., Algorithms, Graphs and Computers, Academic Press, Kyoritsu Shuppan, 1972 (in Japanese)
49. Busacker, R. G. and Saaty, T. L., Finite Graphs and Computers ; An Introduction with Applications, Baihukan, 1970 (in Japanese)
50. Onodera, R., Foundations of Graph Theory, Mathematical Library, No. 6, Morikita Shuppan, 1971 (in Japanese)
51. Onodera, R., Development and Application of Graph Theory, Mathematical Library, No. 30, Morikita Shuppan, 1973 (in Japanese)
52. Onodera, R. and Oorui, H., Fundamental Theory of Electric Circuit, Gakkensha, 1967 (in Japanese)
53. Ore, O., Graphs and Their Uses, New Mathematical Library, No. 5, Kawade-Shobo-Shinsha, 1973 (in Japanese)
54. Kawada, K., Topology, Iwanami Shoten (in Japanese)
55. Kawada, K., Topology, Kisosuugaku-koza 21, Kyoritsu Shuppan (in Japanese)
56. Arnold, B. H., Intuitive Concepts in Elementary Topology, Prentice-Hall, Inc., 1962

57. Wallace, A. H., An Introduction to Algebraic Topology, pergamon Press, 1957, Kyoritsu Shuppan, 1971 (in Japanese)
58. Eilenberg, S. and Steenrod, H., Foundations of Algebraic Topology, Princeton University Press
59. Noguchi, H., Theory of Catastrophe, Kodansha, 1973 (in Japanese)
60. Noguchi, H., Topology ; Appreciation Towards Catastrophe Theory, Nippon-hyoron-sha, 1973 (in Japanese)
61. Homma, T., New Topology ; From Its Foundation to Catastrophe Theory, Blue Backs Series, Kohdansha, 1973 (in Japanese)

Note : References from 59 to 61 were not directly used in this thesis but gave influence on the investigations for graph-theoretical study on Bandwidth Problem.